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Exact solution of the Dirac equation in the presence of pseudoscalar potentials

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Summary. — In the present article we analyze the problem of a relativistic Dirac electron in the presence of the pseudoscalar potentials $\alpha\gamma^5/|z|$, $\alpha\gamma^5/\varrho$, and $\alpha\gamma^5/r$ in Cartesian, cylindrical and spherical coordinates, respectively. After separating variables, we compute the solutions of the Dirac equation and show that there are not bound states of the energy.

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1. – Introduction

In studying the quantum behaviour of relativistic particles in the presence of external fields special attention is paid to the possible existence of bound states, that is states where the energy is discrete and the wave function normalizable. Exact solutions of the Klein-Gordon and the Dirac equations in the presence of external fields cannot always be found in terms of special functions even when the non-relativistic Schrödinger problem is solvable. Perhaps among the most dramatic examples we can mention the two-centre problem, the hydrogen atom in parabolic coordinates, and the Zeeman effect [1], which do not allow the separation of variables in the Dirac equation, and therefore it is not possible to reduce the problem to a system of ordinary differential equations, a necessary step in order to attempt to solve the problem in closed form. A different kind of problem arises when the external field allows a complete separation of variables in the Dirac equation, but the resulting equations are not solvable in terms of special functions. Here we can mention the harmonic-oscillator problem as one of the most representative examples.

Exact solutions of wave equations permit a better understanding of the quantum behaviour of relativistic particles in the presence of external fields. When they exist like for the hydrogen atom, it is possible to analyze the bound states, the influence of

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the spin, and the non-relativistic limit of the energy spectrum. Regretfully, in most of the cases no exact solutions are available and perturbative or variational methods are necessary in order to get some insight into the problem. The search for vector and scalar potential for which the Dirac equation presents bound states is a problem that has motivated a large number of papers. Here we have to mention the existence of potentials which bind particles in the relativistic case but not for the Schrödinger equation [2].

Despite the great body of articles devoted to the discussion of bound states of the Dirac equation in the presence of external fields, most of them deal with electromagnetic interactions on Lorentz scalar potentials [2-5]. Perhaps the only remarkable exception of this situation is the Dirac oscillator by Moshinsky and Szczeparniak [6].

Recently, the problem of separation of variables of the Dirac equation in the presence of scalar and pseudoscalar fields has been discussed by Shishkin [7]. Applying the algebraic method of separation [8, 9], he analyzes the necessary and sufficient conditions allowing a complete separation of variables. A further step in the analysis of pseudoscalar fields in the framework of the Dirac equation could be to obtain exact solutions and search for bound states when only a pseudoscalar potential is present. Among the possible candidates for being first discussed we have the potential γ^5/r in spherical coordinates. The reasons for that become evident if we look at the results obtained when we deal with the Coulomb potential or the scalar $1/r$ potential and their strange behaviour when we discuss the one-dimensional case. Also we have that the Dirac equation with the pseudoscalar γ^5/r potential is completely separable and exact solutions are obtained in closed form.

It is the purpose of the present article to solve the Dirac equation in the presence of the pseudoscalar potential γ^5/r in spherical coordinates, γ^5/ϱ in cylindrical coordinates, and $\gamma^5/|z|$ in Cartesian coordinates. We show that no bound states are present in any of the three configurations. Therefore, pseudoscalar potentials with a dependence inversely proportional to the radial variable do not bind particles.

The article is structured as follows. In sect. **2** we solve the Dirac equation in Cartesian coordinates in the presence of the pseudoscalar potential $\gamma^5/|z|$. In sect. **3** the Dirac equation is solved in the potential γ^5/ϱ. In sect. **4** the Dirac equation is solved in the presence of the potential γ^5/r in spherical coordinates. Finally, some concluding remarks are presented in sect. **5**.

2. – Dirac equation in Cartesian coordinates

In this section we solve the Dirac equation in Cartesian coordinates in the presence of the pseudoscalar potential

$$V = \alpha\gamma^5/|z| \,, \tag{2.1}$$

where α is a real constant. Then the Dirac equation reads

$$\left(\gamma^0\partial_t + \gamma^1\partial_x + \gamma^2\partial_y + \gamma^3\partial_z + m + i\gamma^0\gamma^1\gamma^2\gamma^3\,\frac{\alpha}{|z|}\right)\Psi = 0\,, \tag{2.2}$$

where we have followed the Jauch and Rohrlich [10] convention on $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ and we have inserted i before γ^5 in (2.2) order to make the Dirac Hamiltonian Hermitian.

After applying the algebraic method of separation of variables, eq. (2.2) can be re-written in terms of first-order commuting operators $\widehat{K}_1$, $\widehat{K}_2$ as follows:

$$(\widehat{K}_1 + \widehat{K}_2)\,\Phi = 0\,, \qquad \widehat{K}_2\,\Phi = k\Phi = -\widehat{K}_1\,\Phi\,, \qquad \Phi = \gamma^1\gamma^2\Psi\,, \tag{2.3}$$

where

$$\widehat{K}_1 = (\gamma^1\partial_x + \gamma^2\partial_y + m)\,\gamma^1\gamma^2\,, \tag{2.4}$$

$$\widehat{K}_2 = (\gamma^0\partial_t + \gamma^3\partial_z + m + i\gamma^0\gamma^1\gamma^2\gamma^3 V(z))\,\gamma^1\gamma^2\,. \tag{2.5}$$

Since the Dirac equation (2.2) commutes with the operators $i\partial_t$, $-i\partial_x$ and $-i\partial_y$ we can write the spinor Φ as follows:

$$\Phi = \Phi_0(z)\exp\left[i(k_x x + k_y y - Et)\right]. \tag{2.6}$$

Choosing to work in the Dirac matrices' representation [10]

$$\gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & +i \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \sigma^1 \\ \sigma^1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}, \tag{2.7}$$

we obtain that the equation $\widehat{K}_2\,\Phi = k\Phi$ can be written in terms of Pauli matrices as follows:

$$(-E\sigma^3 + ik)\,\Phi_+ + \left(d_z - \sigma^3\frac{\alpha}{|z|}\right)\Phi_- = 0\,, \tag{2.8}$$

$$(E\sigma^3 + ik)\,\Phi_- + \left(d_z + \sigma^3\frac{\alpha}{|z|}\right)\Phi_+ = 0\,, \tag{2.9}$$

where Φ_+ and Φ_- are, respectively, the upper and lower components of the bispinor Φ. From eqs. (2.8) and (2.9) we obtain

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} \mp \frac{\alpha}{z}\right)\xi_{\mp} + (ik \mp E)\,\xi_{\pm} = 0\,, \tag{2.10}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}z} \mp \frac{\alpha}{z}\right)\eta_{\pm} + (ik \mp E)\,\eta_{\mp} = 0 \tag{2.11}$$

with

$$\Phi_+ = \begin{pmatrix} \xi_+ \\ \eta_+ \end{pmatrix}, \qquad \Phi_- = \begin{pmatrix} \xi_- \\ \eta_- \end{pmatrix}, \tag{2.12}$$

where, for the sake of simplicity, we have omitted the absolute value of $|z|$ in the potential. This is equivalent to considering that eqs. (2.10) and (2.11) are valid for positive values of the variable z. The negative branch can be obtained by substituting α by $-\alpha$ in eqs. (2.10)-(2.11) and, consequently, in all the results presented in this section. The divergent behaviour of the potential at the origin does not need to be considered because the left $(z<0)$ and right $(z>0)$ regions are independent [11, 12].

Then, from (2.10) and (2.11) we get

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{\alpha(\alpha \pm 1)}{z^2} + (E^2 + k^2)\right) \xi_{\pm} = 0\,, \tag{2.13}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2} - \frac{\alpha(\alpha \pm 1)}{z^2} + (E^2 + k^2)\right) \eta_{\mp} = 0\,. \tag{2.14}$$

The solution of eq. (2.13) (and therefore of eq. (2.14)) can be readily obtained in terms of Bessel functions $Z_\mu(x)$ [13],

$$\xi_{\pm} = c_{\pm} \sqrt{z}\, Z_{\alpha \pm 1/2}(\sqrt{E^2 + k^2} z)\,, \tag{2.15}$$

where the relation between the constants c_+ and c_- can be obtained from eq. (2.10) and the recurrence relation among the Bessel functions. The ratio c_+/c_- reads

$$\frac{c_+}{c_-} = -\frac{ik + E}{\sqrt{E^2 + k^2}}\,. \tag{2.16}$$

In order to determine the structure of the spinor solution Φ, as well as the value of the constant of separation k, we proceed to substitute the gamma representation (2.7) into the eigenvalue equation $\widehat{K}_1 \Phi + k\Phi = 0$. Then we obtain

$$(m\sigma^3 - ik)\, \Phi_{+,\,-} + (k_x \sigma^2 - k_y \sigma^1)\, \Phi_{-,\,+} = 0 \tag{2.17}$$

and the constant of separation k satisfies the equation

$$k^2 + m^2 + k_x^2 + k_y^2 = 0\,. \tag{2.18}$$

Finally, we find that the spinor Φ in the matrix representation (2.7) takes the form

$$\Phi = \begin{pmatrix} ((ik_x + k_y)/(m - ik))\, \eta_- \\ \eta_+ \\ ((ik_x + k_y)/(m - ik))\, \eta_+ \\ \eta_- \end{pmatrix} \exp[i(k_x x + k_y y - Et)]\,; \tag{2.19}$$

in the case when $k_x = k_y = 0$ we have that eigenvalue k can take the values $\pm im$ having the spinor solution (2.19) sense when $k = -im$. From (2.17) it is straightforward to obtain the structure of Φ when $k = im$. As to the possible existence of bound states of the energy for eq. (2.2), we have that solution (2.15) shows this is not the case. In fact, for $E^2 + k^2 < 0$ the solution of (2.13) can expressed in terms of McDonald's functions $K_\mu(x)$ [13],

$$\xi_{\pm} = d_{\pm} \sqrt{z}\, K_{\alpha \pm 1/2}(\sqrt{m^2 + k_x^2 + k_y^2 - E^2} z) \tag{2.20}$$

which go to zero as z goes to infinity, but diverges at the origin when $\alpha > 0$ becoming the wave function non-normalizable, and therefore only states with $E^2 + k^2 > 0$ are possible. The case $\alpha < 0$ with $E^2 + k^2 < 0$ makes the particle fall to the origin $z = 0$ [14].

The absence of bound states for the Dirac equation with the pseudoscalar potential (2.1) has its analogous in the one-dimensional hydrogen atom [12], where a Coulomb field does not confine the electron.

3. – Dirac equation in cylindrical coordinates

In this section we solve the Dirac equation in the presence of the pseudoscalar field

$$V = \gamma^5 \frac{\alpha}{\varrho}\,, \tag{3.1}$$

where $\varrho = \sqrt{x^2 + y^2}$. Since the potential (3.1) is not separable in Cartesian coordinates, we proceed to tackle the problem in cylindrical coordinates. The Dirac equation expressed in the rotating diagonal tetrad reads [8, 9, 15]

$$\left(\gamma^0 \partial_t + \gamma^1 \partial_\varrho + \frac{\gamma^2}{\varrho} \partial_\vartheta + \gamma^3 \partial_z + i\gamma^0\gamma^1\gamma^2\gamma^3 \frac{\alpha}{\varrho} + m\right)\Psi_{\text{diag}} = 0\,, \tag{3.2}$$

where we have been able to eliminate the contribution of the spinor connections as well as the coordinate dependence of the gamma matrices by the help of the similarity transformation S,

$$S = \frac{1}{\sqrt{\varrho}} \exp\left[-\frac{\vartheta}{2}\gamma^1\gamma^2\right] \tag{3.3}$$

which acts on Ψ as follows:

$$\Psi = \Psi_{\text{Cart}} = S^{-1}\Psi_{\text{diag}}\,. \tag{3.4}$$

Equation (3.2) can be written as a sum of two commuting first-order differential operators $\widehat{K}_1$ and $\widehat{K}_2$ as follows:

$$(\widehat{K}_1 + \widehat{K}_2)\,\Phi = 0\,, \qquad \Phi = \gamma^0\gamma^3\Psi_{\text{diag}}\,, \tag{3.5}$$

where $\widehat{K}_1$ and $\widehat{K}_2$ are the linear operators

$$\widehat{K}_1 = (\gamma^0\partial_t + \gamma^3\partial_z + m)\,\gamma^0\gamma^3\,, \tag{3.6}$$

$$\widehat{K}_2 = \gamma^1\gamma^0\gamma^3\partial_\varrho + \frac{1}{\varrho}\gamma^2\gamma^0\gamma^3\partial_\vartheta + i\gamma^1\gamma^2 V(\varrho) \tag{3.7}$$

that satisfy the eigenvalue equation $\widehat{K}_1\Phi = k\Phi = -\widehat{K}_2\Phi$. Since eq. (3.2) commutes with the operators $i\partial_t - i\partial_\vartheta$ and $-i\partial_z$ with eigenvalues E, k_ϑ and k_z, respectively, we can write the spinor Φ as follows:

$$\Phi = \Phi_0(\varrho)\exp\left[i(k_\vartheta\vartheta + k_z z - Et)\right]. \tag{3.8}$$

Equation (3.6) is useful for establishing the structure of the bispinor Φ, and the value of the constant of separation k, but does not give information about the radial behaviour of the wave function and the possible existence of bound states. Therefore we proceed to solve the eigenvalue equation $\widehat{K}_1\Phi = k\Phi$. Since in (3.7) there are three terms containing matrices anticommuting with each other, it is not possible to find a representation of the gamma matrices allowing to decouple the equation $k\Phi + \widehat{K}_2\Phi = 0$ in terms of two-component spinors. Therefore, it is convenient to introduce the auxiliary bispinor Φ given by the expression

$$\Phi = \left(\gamma^1 \gamma^0 \gamma^3 \partial_\varrho + \frac{ik_\vartheta}{\varrho} \gamma^2 \gamma^0 \gamma^3 + i\gamma^1 \gamma^2 \frac{\alpha}{\varrho} - k\right) \Theta . \tag{3.9}$$

Then, after substituting eq. (3.9) into the eigenvalue equation for $\widehat{K}_2$, we obtain

$$\left(\frac{d^2}{d\varrho^2} - \frac{1}{\varrho^2}(k_\vartheta^2 + \alpha^2) - \frac{1}{\varrho^2}(ik_\vartheta \gamma^1 \gamma^2 + i\alpha\gamma^0 \gamma^3 \gamma^2) - k^2\right) \Theta ; \tag{3.10}$$

using the representation (2.7) for the Dirac matrices, we are able to reduce eq. (3.10) to the form

$$\left(\frac{d^2}{d\varrho^2} - \frac{1}{\varrho^2}(k_\vartheta^2 - \alpha^2) + \frac{1}{\varrho^2}(k_\vartheta \sigma^3 \pm i\alpha\sigma^1) - k^2\right) \Theta_{+,\,-}, \quad \text{with } \Theta = \begin{pmatrix} \Theta_+ \\ \Theta_- \end{pmatrix}; \tag{3.11}$$

eq. (3.11) can be reduced to a diagonal form with the help of the operator Q:

$$Q^{-1}(k_\vartheta \sigma^3 + i\alpha\sigma^1)\, Q = \sqrt{k_\vartheta^2 - \alpha^2}\sigma^3 , \qquad Q(\alpha) = \exp\left[\frac{1}{2} \operatorname{tgh}^{-1}\left(\frac{\alpha}{k_\vartheta}\right)\sigma^2\right], \tag{3.12}$$

whence we find that the operator S acting on Θ in (3.10) reads

$$S = \begin{pmatrix} Q(\alpha) = & 0 \\ 0 & Q(-\alpha) \end{pmatrix}, \qquad S^{-1}\Theta = \widetilde{\Theta} = \begin{pmatrix} \widetilde{\Theta}_+ \\ \Theta_- \end{pmatrix} \tag{3.13}$$

from which we obtain

$$\left(\frac{d^2}{d\varrho^2} - \frac{\lambda(\lambda \mp 1)}{\varrho^2} - k^2\right) \widetilde{\Theta}_{+,\,-} , \tag{3.14}$$

$$\lambda = \sqrt{k_\vartheta^2 - \alpha^2} \tag{3.15}$$

from the algebraic equation $\widehat{K}_1 \Phi = k\Phi$ we have that $k^2 = m^2 + K_z^2 - E^2$, and, consequently, the solution of (3.14) can be expressed in terms of Bessel functions $J_v(x)$ as follows:

$$\widetilde{\Theta}_+ = c_\pm \varrho^{1/2} J_{\lambda \mp 1/2}(\sqrt{E^2 - m^2 - k_z^2}\varrho) , \tag{3.16}$$

where c_+ and c_- are constants. From eq. (3.16) we see that free states are normalizable. For $E^2 - m^2 - k_z^2 < 0$ the solutions of (3.14) can be expressed in terms of McDonald's functions like equation (2.20) in sect. **2**. Then, no bound states are possible for the Dirac equation in the presence of the pseudoscalar potential (3.1).

4. – Dirac equation in spherical coordinates

In this section we are interested in solving the Dirac equation in the presence of the potential

$$V(r) = \gamma^5 \frac{\alpha}{r} , \tag{4.1}$$

where r is the distance to the origin. Then, it is convenient to work in the present case in spherical coordinates. The Dirac equation expressed in the rotating diagonal gauge with the potential (4.1) reads

$$\left(\gamma^0 \partial_t + \gamma^1 \partial_r + \frac{\gamma^2}{r} \partial_\vartheta + \frac{\gamma^3}{r \sin\vartheta} \partial_\varphi + i\gamma^0\gamma^1\gamma^2\gamma^3 \frac{\alpha}{r} + m\right) \Psi_d = 0 , \tag{4.2}$$

where in the present case the matrix transformation S, which relates the Dirac spinor Ψ, in the Cartesian tetrad gauge, with Ψ_d reads [16]

$$S\Psi_d = \Psi , \qquad S = \frac{1}{r(\sin\vartheta)^{1/2}} \exp\left[-\frac{\varphi}{2}\gamma^1\gamma^2\right] \exp\left[-\frac{\vartheta}{2}\gamma^3\gamma^1\right] N \tag{4.3}$$

with

$$N = \frac{1}{2}(\gamma^1\gamma^2 + \gamma^2\gamma^3 + \gamma^3\gamma^1 + 1), \tag{4.4}$$

in this way, we avoid working with additional terms, arising from the contribution of the spinor connections.

Now, we can proceed as in the previous sections and, after separating the angular variables in eq. (4.2), we arrive at

$$\left[(\gamma^0 \partial_t + \gamma^1 \partial_r + m)\,\gamma^0\gamma^1 + \frac{i\lambda}{r}\right] \Phi , \tag{4.5}$$

$$\left[\left(\gamma^2 \partial_\vartheta + \frac{\gamma^3}{\sin\vartheta} \partial_\varphi\right)\gamma^0\gamma^1 + i\alpha\gamma^2\gamma^3\right] \Phi = i\lambda\Phi , \tag{4.6}$$

where λ is a constant of separation and $\Phi = \gamma^0\gamma^1\Psi$. Notice that the contribution of the pseudoscalar potential goes into eq. (4.6) because of its particular dependence on r. Any other functional dependence on the radial variable would require to include the potential into the separating operator corresponding to the radial dependence, but in that case the separation is only possible in the massless case.

Equation (4.6) can be written as

$$(\widehat{K} + \alpha\gamma^2\gamma^3)\,\Phi = \lambda\Phi , \tag{4.7}$$

where $\widehat{K}$ is the Brill and Wheeler [17] angular-momentum operator, which anticommutes with the matrix $\gamma^2\gamma^3$,

$$\widehat{K} = -i\left(\gamma^2 \partial_\vartheta + \frac{\gamma^3}{\sin\vartheta} \partial_\varphi\right)\gamma^0\gamma^1 , \qquad \{\widehat{K}, \gamma^2\gamma^3\}_+ = 0 . \tag{4.8}$$

From eq. (4.7) it is not difficult to obtain

$$k^2 + \alpha^2 = \lambda^2 , \tag{4.9}$$

where k is the eigenvalue of the momentum operator $\widehat{K}$. Since the potential does not depend on time, nor on the azimuthal angle φ, we have that the bispinor Φ can be written as

$$\Phi = \Phi_0(r, \vartheta) \exp[i(k_\varphi \varphi - Et)], \tag{4.10}$$

then eq. (4.5), written in the representation (2.7) of the Dirac matrices, reads

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\lambda}{r}\right)\Phi_1+(E+m)\,\sigma^1\,\Phi_2=0\,, \tag{4.11}$$

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}+\frac{\lambda}{r}\right)\Phi_2-(E+m)\,\sigma^1\,\Phi_1=0\,, \tag{4.12}$$

where Φ_1 and Φ_2 are, respectively, the upper and lower components of the bispinor Φ. Substituting (4.12) into (4.11) and vice versa we find

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{\lambda(\lambda\pm 1)}{r^2}+(E^2-m^2)\right)\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}=0\,, \tag{4.13}$$

eq. (4.13) has the same form as eq. (2.13) in sect. **2**. Consequently, only free states for the relativistic Dirac electron are possible in the presence of the pseudoscalar field (4.1). The solutions of eq. (4.13) can be expressed in terms of Bessel functions,

$$\Phi_{1,\,2}=c_{1,\,2}\,r^{1/2}J_{\lambda\pm 1/2}(\sqrt{E^2-m^2}\,r)\,. \tag{4.14}$$

5. – Discussion of the results

The results obtained in the present article show that the inclusion in the Dirac equation of pseudoscalar potentials with a functional dependence inversely proportional to the distance is not enough for obtaining bound states of the energy. Then three cases discussed here also show that the pseudoscalar nature of the potential dramatically changes the behaviour of the wave functions according to the inclusion of Lorentz scalar potentials with the same functional dependence.

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