

CREATION OF DIRAC PARTICLES IN THE PRESENCE OF A CONSTANT ELECTRIC FIELD IN AN ANISOTROPIC BIANCHI I UNIVERSE

VÍCTOR M. VILLALBA* and WALTER GREINER

*Institut für Theoretische Physik, Universität Frankfurt,
D-60054 Frankfurt am Main, Germany*

**villalba@th.physik.uni-frankfurt.de*

Received 4 September 2002

In this article we compute the density of Dirac particles created by a cosmological anisotropic Bianchi I universe in the presence of a constant electric field. We show that the particle distribution becomes thermal when one neglects the electric interaction.

Keywords: Particle creation; cosmology.

PACS Nos.: 03.65.Pm, 04.62.+v

1. Introduction

During the last decade a great effort has been made in understanding quantum processes in strong fields. Quantum field theory in the presence of strong fields is in general a theory associated with unstable vacua. The vacuum instability leads to many interesting features, among them particle creation is perhaps the most interesting nonperturbative phenomenon. After the publication of the articles by Parker^{1,2} and Zeldovich,³ the study of quantum effects in cosmology became a very active research field. In the last three decades a large body of papers has been published on the problem of vacuum effects in isotropic and homogeneous gravitational backgrounds, mainly in de Sitter and Robertson–Walker models, and only a few try to discuss quantum processes in anisotropic Universes.

In order to analyze the mechanism of particle creation in cosmological backgrounds we have at our disposal different techniques such as the adiabatic approach,^{1,4} the Feynman path integral method,⁵ the Hamiltonian diagonalization technique,^{6,7} as well as the Green function approach.⁸

The study of quantum effects in gravitational backgrounds with initial singularities presents an additional difficulty. The techniques commonly applied in order

*On leave of absence from Centro de Física, Instituto Venezolano de Investigaciones Científicas, IVIC, Apdo 21827, Caracas 1020-A, Venezuela. Alexander von Humboldt Fellow

to define particle states are based on the existence of a timelike Killing vector or an asymptotically static metric.⁹ A different approach is needed to circumvent the problem related to the initial singularity. In this direction, the Feynman path-integral method has been applied to the quantization of a scalar field moving in the Chitre–Hartle Universe.^{5,11} This model has a curvature singularity at $t = 0$, and it is perhaps the best known example where a time singularity appears and consequently any adiabatic prescription in order to define particle states fails. A spin 1/2 extension has been considered by Sahni.¹²

A different approach to the problem of classifying single particle states on curved spaces, is based on the idea of diagonalizing the Hamiltonian. This technique permits one to compute the mean number of particles produced by a singular cosmological model, and in particular by the Chitre–Hartle Universe.⁵

An interesting scenario for discussing particle creation processes is the anisotropic universe associated with the metric

$$ds^2 = -dt^2 + t^2(dx^2 + dy^2) + dz^2. \quad (1)$$

The line element (1) presents a space-like singularity at $t = 0$. The scalar curvature is $R = 2/t^2$, and consequently, the adiabatic approach⁹ cannot be applied in order to define particle states. Creation of scalar particles in the anisotropic universe (1) was originally computed by Nariai with the help of Feynman propagators, obtaining as a result that the creation process occurs in accordance with the black-body thermal law in a two-dimensional hypersurface related to the anisotropic cosmic expansion. With the help of the Hamiltonian diagonalization method,^{6,13,14} Bukhbinder⁷ could compute the rate of scalar particles produced in the space with the metric (1), obtaining as a result, a Bose–Einstein distribution. In Ref. 15 this result has been extended to include a time-dependent electric field. More recently,^{16,17} a quasiclassical approach has been applied to compute the rate of scalar as well as Dirac particles in the cosmological universe associated with the metric (1).

The introduction of an external electric field permits one to consider an additional source of quantum processes. The density of particles created by an intense electric field was first calculated by Schwinger,¹⁸ different authors^{6,19} have discussed this problem. Pair creation of scalar particles by a constant electric field in a 2+1 de Sitter cosmological universe has been analyzed by Garriga.²⁰ Quantum effects associated with scalar and spinor particles in a quasi-Euclidean cosmological model with a constant electric field are discussed by Bukhbinder and Odintsov.²¹ It is worth mentioning that the presence of primordial electric fields could enhance the particle creation mechanism and also produces deviations from the thermal spectrum.

The purpose of this article is to discuss the production of Dirac particles in the anisotropic cosmological background associated with the line element (1) in the presence of a constant electric field. In order to compute the rate of particles created we apply a quasiclassical approach that has been successfully applied in different scenarios.^{17,22–24} The idea behind the method is the following: First, we solve the relativistic Hamilton–Jacobi equation and, looking at its solutions, we

identify positive and negative frequency modes. Second, we solve the Dirac equation and, after comparing with the results obtained for the quasiclassical limit, we identify the positive and negative frequency states.

The paper is structured as follows. In Sec. 2 we solve the relativistic Hamilton–Jacobi equation and compute the quasiclassical energy modes. In Sec. 3, after separating variables, we solve the Dirac equation and obtain the density of Dirac particles created. The discussion of the results and final remarks are presented in Sec. 4.

2. Hamilton–Jacobi Equation

The relativistic Hamilton–Jacobi equation can be written as

$$g^{\alpha\beta} \left(\frac{\partial S}{\partial x^\alpha} - eA_\alpha \right) \left(\frac{\partial S}{\partial x^\beta} - eA_\beta \right) + m^2 = 0, \tag{2}$$

where here and elsewhere we adopt the convention $c = 1$ and $\hbar = 1$.

The vector potential A_α

$$A_\alpha = (0, 0, 0, -Et), \tag{3}$$

corresponds to a constant electric field $E\hat{k}$. The corresponding invariants $F^{\mu\nu}F_{\mu\nu} = -2E^2$ and $F^{\mu\nu*}F_{\mu\nu} = 0$ indicate that there is no magnetic field. Since the metric $g_{\alpha\beta}$ associated with the line element (1) only depends on t , the function S can be separated as

$$S = F(t) + k_x x + k_y y + k_z z. \tag{4}$$

Substituting (4) into (2) we obtain

$$\dot{F}^2 = \frac{k_x^2 + k_y^2}{t^2} + (k_z + eEt)^2 + m^2. \tag{5}$$

The solution of Eq. (5) presents the following asymptotic behavior:

$$\lim_{t \rightarrow \infty} F = \pm \frac{1}{2} t \sqrt{e^2 E^2 t^2 - m^2} \mp \frac{m^2}{2eE} \log (eEt + \sqrt{e^2 E^2 t^2 - m^2}), \tag{6}$$

$$\Phi = e^{iS} \rightarrow C e^{\pm \frac{i}{2} eEt^2} (eEt)^{\mp \frac{im^2}{2eE}} \tag{7}$$

as $t \rightarrow \infty$, and

$$\lim_{t \rightarrow 0} F = \pm \sqrt{(k_x^2 + k_y^2) \log t}, \quad \Phi = e^{iS} \rightarrow C t^{\pm i \sqrt{k_x^2 + k_y^2}}, \tag{8}$$

as $t \rightarrow 0$, that is, in the initial singularity. Notice that the time dependence of the relativistic wave function is obtained via the exponential operation $\Phi \rightarrow \exp(iS)$. Here the behavior of positive and negative frequency states is selected depending on the sign of the operator $i\partial_t$. Positive frequency modes will have positive eigenvalues and for negative frequency states we will have negative eigenvalues. Then in Eqs. (6)–(8), upper signs are associated with negative frequency values and the

lower signs correspond to positive frequency states. After making this identification we proceed to analyze the solutions of the Dirac equation in the cosmological background (1).

3. Solution of the Dirac Equation

The covariant Dirac equation in curved space in the presence of electromagnetic fields can be written as follows:

$$\{\gamma^\mu(\partial_\mu - \Gamma_\mu - ieA_\mu) + m\}\Psi = 0, \tag{9}$$

where the curved gamma γ^μ matrices satisfy the anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and the spinor connections Γ_μ are

$$\Gamma_\mu = \frac{1}{4}g_{\lambda\alpha} \left[\left(\frac{\partial b_\nu^\beta}{\partial x^\mu} \right) a_\beta^\alpha - \Gamma_{\nu\mu}^\alpha \right] s^{\lambda\nu}, \tag{10}$$

where

$$s^{\lambda\nu} = \frac{1}{2}(\gamma^\lambda\gamma^\nu - \gamma^\nu\gamma^\lambda). \tag{11}$$

The matrices $b_\alpha^\beta, a_\beta^\alpha$ establish the connection between the curved γ^μ and Minkowski $\tilde{\gamma}^\mu$ Dirac matrices as follows:

$$\gamma_\mu = b_\mu^\alpha \tilde{\gamma}_\alpha, \quad \gamma^\mu = a_\beta^\mu \tilde{\gamma}^\beta \tag{12}$$

and

$$\tilde{\gamma}^\lambda \tilde{\gamma}^\nu + \tilde{\gamma}^\nu \tilde{\gamma}^\lambda = 2\eta^{\lambda\nu}. \tag{13}$$

Since the line element (1) is diagonal, we choose to work in the diagonal tetrad

$$\gamma^\mu = \sqrt{|g^{\mu\mu}|} \tilde{\gamma}^\mu, \quad \text{no sum.} \tag{14}$$

Substituting (14) into (10), we obtain

$$\Gamma_1 = \frac{1}{2}\tilde{\gamma}^0\tilde{\gamma}^1, \quad \Gamma_2 = \frac{1}{2}\tilde{\gamma}^0\tilde{\gamma}^2, \quad \Gamma_3 = 0, \quad \Gamma_0 = 0 \tag{15}$$

and the Dirac equation (9) takes the form

$$\left\{ \tilde{\gamma}^0 \frac{\partial}{\partial t} + \frac{1}{t} \left(\tilde{\gamma}^1 \frac{\partial}{\partial x} + \tilde{\gamma}^2 \frac{\partial}{\partial y} \right) + \tilde{\gamma}^3 \left(\frac{\partial}{\partial z} + ieEt \right) + m \right\} \Psi_0 = 0, \tag{16}$$

where $\Psi_0 = t\Psi$. The factor t was introduced in order to cancel the contribution due to the spinor connections (15). Equation (16) can be written as a sum of two first-order commuting differential operators as follows^{25,26}:

$$(\hat{K}_1 + \hat{K}_2)\Phi = 0, \tag{17}$$

$$\hat{K}_2\Phi = k\Phi = -\hat{K}_1\Phi, \tag{18}$$

where the spinor Φ is related to Ψ_0 via the equation

$$\tilde{\gamma}^3\tilde{\gamma}^0\Psi_0 = \Phi, \tag{19}$$

and k is a separation constant. The operators \hat{K}_1 and \hat{K}_2 read

$$\hat{K}_1 = t \left[\gamma^3 \frac{\partial}{\partial t} + \gamma^0 \left(\frac{\partial}{\partial z} - ieEt \right) + m\gamma^3\gamma^0 \right], \tag{20}$$

$$\hat{K}_2 = \left(\tilde{\gamma}^1 \frac{\partial}{\partial x} + \tilde{\gamma}^2 \frac{\partial}{\partial y} \right) \tilde{\gamma}^3 \tilde{\gamma}^0. \tag{21}$$

Since Eq. (16) commutes with $-i\nabla$, the spinor Φ can be written as $\Phi = \Phi_0 \exp(i(k_x x + k_y y + k_z z))$. Choosing to work in the following representation of the Dirac matrices

$$\tilde{\gamma}^0 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & i\sigma^1 \end{pmatrix}, \quad \tilde{\gamma}^1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tag{22}$$

$$\tilde{\gamma}^2 = \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix}, \quad \tilde{\gamma}^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}$$

we obtain that $\hat{K}_2\Phi = k\Phi$ reduces to an algebraic equation that permits one to determine the relation between the components of the bispinor Φ

$$\Phi_0 = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} \Phi_1 \\ \frac{k_x\sigma^2}{ik_y - k} \Phi_1 \end{pmatrix}, \tag{23}$$

where eigenvalue k is

$$k = i\sqrt{k_x^2 + k_y^2}. \tag{24}$$

Using the representation (22) and taking into account the spinor structure (23) we obtain that, for $k_z = 0$, Eq. (20) reduces to the system of equations

$$\frac{d\varphi_1}{dt} + \frac{k}{t}\varphi_1 + (m + ieEt)\varphi_2 = 0, \tag{25}$$

$$-\frac{d\varphi_2}{dt} + \frac{k}{t}\varphi_2 + (m - ieEt)\varphi_1 = 0. \tag{26}$$

In this way, we have reduced the problem of solving Eq. (16) to that of finding the solution of Eqs. (25) and (26). From (25) and (26) we obtain the following second-order differential equation

$$\frac{d^2\psi_2}{dt^2} + \left(-\frac{k^2 - 2k + 3/4}{t^2} + m^2 + e^2E^2t^2 \right) \psi_2 = 0, \tag{27}$$

where we have introduced the variable $\psi_2 = t^{-1/2}\varphi_2$ and have neglected the mass in the first-order variation of φ_2 . The solution of (27) can be expressed in terms of Whittaker functions as

$$\psi_2 = C_1 M_{\lambda,\mu}(ieEt^2) + C_2 W_{\lambda,\mu}(ieEt^2), \tag{28}$$

where

$$\lambda = -\frac{im^2}{4eE}, \quad \mu = \frac{k}{2} - \frac{1}{2}, \tag{29}$$

and C_1 and C_2 are arbitrary constants. Analogously, after introducing the new variable $\psi_1 = t^{-1/2}\phi_1$, we have that the equation for ψ_1 is

$$\frac{d^2\psi_1}{dt^2} + \left(-\frac{k^2 + 2k + 3/4}{t^2} + m^2 + e^2 E^2 t^2 \right) \psi_1 = 0 \tag{30}$$

with solutions

$$\psi_1 = C_3 M_{\lambda, \mu+1}(ieEt^2) + C_4 W_{\lambda, \mu+1}(ieEt^2), \tag{31}$$

where the Whittaker functions $W_{k, \mu}(z)$ and $M_{k, \mu}(z)$ can be expressed in terms of confluent hypergeometric functions²⁷ as follows:

$$M_{k, \mu}(z) = e^{-z/2} z^{1/2+\mu} M\left(\frac{1}{2} + \mu - k, 1 + 2\mu, z\right), \tag{32}$$

$$W_{k, \mu}(z) = e^{-z/2} z^{1/2+\mu} W\left(\frac{1}{2} + \mu - k, 1 + 2\mu, z\right). \tag{33}$$

In order to construct the positive and negative frequency modes we use the asymptotic behavior of the hypergeometric functions (32) and (33) and compare the solutions of Eq. (27) with (6) and (8), obtained after solving the Hamilton–Jacobi equation. Using this procedure and looking at the asymptotic behavior of $M_{k, \mu}(z)$ as $z \rightarrow 0$

$$M_{k, \mu}(z) \sim e^{-z/2} z^{1/2+\mu} \tag{34}$$

and using the fact that all the coefficients in Eq. (27) are real, we obtain that the positive and negative frequency solutions as $t \rightarrow 0$ are

$$\begin{aligned} \psi_0^+ &= \mathfrak{C}_0^+ M_{\lambda, \mu}(ieEt^2), \\ \psi_0^- &= (\mathfrak{C}_0^+ M_{\lambda, \mu}(ieEt^2))^* = \mathfrak{C}_0^+ (-1)^{-\bar{\mu}+1/2} M_{\lambda, -\mu}(ieEt^2), \end{aligned} \tag{35}$$

where \mathfrak{C}_0^+ is a normalization constant. Analogously, looking at the behavior of $W_{k, \mu}(z)$ as $|z| \rightarrow \infty$

$$W_{k, \mu}(z) \sim e^{-z/2} z^k, \tag{36}$$

we have that the corresponding positive and negative frequency modes as $t \rightarrow +\infty$ are

$$\psi_\infty^+ = \mathfrak{C}_\infty^+ W_{k, \mu}(ieEt^2), \quad \psi_\infty^- = \mathfrak{C}_\infty^- W_{-k, \mu}(-ieEt^2), \tag{37}$$

where \mathfrak{C}_∞^+ and \mathfrak{C}_∞^- are normalization constants.

Using the asymptotic expressions

$$\lim_{a \rightarrow \infty} M(a, b, -z/a)/\Gamma(b) = z^{\frac{1}{2}-\frac{b}{2}} J_{b-1}(2\sqrt{z}), \tag{38}$$

$$\lim_{a \rightarrow \infty} U(a, b, -z/a)\Gamma(1+a-b) = \mp i\pi e^{\pi ib} z^{\frac{1}{2}-\frac{b}{2}} H_{b-1}^{(1,2)}(2\sqrt{z}), \tag{39}$$

we obtain that, as $E \rightarrow 0$, the positive and negative frequency modes (35) and (37) reduce to those obtained through using the diagonalization method⁷ as well as with the help of the semiclassical approach.¹⁶

The positive frequency mode as $t \rightarrow 0$ can be expressed in terms of the positive ψ_∞^+ and negative ψ_∞^- frequency modes via the Bogoliubov transformation

$$\psi_0^+ = \alpha\psi_\infty^+ + \beta\psi_\infty^- . \tag{40}$$

The Whittaker function $M_{k,\mu}(z)$ can be expressed in terms of $W_{k,\mu}(z)$ as²⁷

$$\begin{aligned} M_{k,\mu}(z) &= \frac{\Gamma(2\mu + 1)}{\Gamma(\mu - k + \frac{1}{2})} e^{-i\pi k} W_{-k,\mu}(-z) \\ &+ \frac{\Gamma(2\mu + 1)}{\Gamma(\mu + k + \frac{1}{2})} e^{-i\pi(k-\mu-\frac{1}{2})} W_{k,\mu}(z) . \end{aligned} \tag{41}$$

Using the expression (41), we have that the negative frequency solution ψ_0^- can be written in terms of ψ_∞^+ and ψ_∞^- as follows:

$$\psi_0^- = \frac{\Gamma(2\mu + 1)}{\Gamma(\mu - \lambda + \frac{1}{2})} e^{-i\pi k} \psi_\infty^- + \frac{\Gamma(2\mu + 1)}{\Gamma(\frac{1}{2} + \mu + \lambda)} (-1)^{-1/4} e^{-i\pi\mu(\lambda-\mu-\frac{1}{2})} (\psi_\infty^-)^* , \tag{42}$$

where we have made use of the property $W_{-k,\mu}(-z) = (W_{k,\mu}(z))^*$.

Since we have been able to obtain single particle states in the vicinity of $t = 0$ as well as in the asymptote $t \rightarrow \infty$, we can compute the density of particles created by the gravitational field, with the help of the Bogoliubov coefficients.^{6,9} From (42) and using the fact that $\psi_0^- = \alpha\psi_\infty^- + \beta(\psi_\infty^-)^*$, we obtain,

$$\frac{|\beta|^2}{|\alpha|^2} = e^{2i\pi\mu} \frac{|\Gamma(\frac{1}{2} + \mu - \lambda)|^2}{|\Gamma(\frac{1}{2} + \mu + \lambda)|^2} . \tag{43}$$

Substituting into (43) the values for μ and k we obtain

$$\frac{|\beta|^2}{|\alpha|^2} = e^{-\pi k_\perp} \frac{\left(\frac{k_\perp}{2} - \frac{m^2}{4eE}\right) \sinh\left(\frac{\pi k_\perp}{2} - \frac{\pi m^2}{4eE}\right)}{\left(\frac{k_\perp}{2} + \frac{m^2}{4eE}\right) \sinh\left(\frac{\pi k_\perp}{2} + \frac{\pi m^2}{4eE}\right)} , \tag{44}$$

where we have used the relation²⁸

$$|\Gamma(iy)|^2 = \frac{\pi}{y \sinh \pi y} . \tag{45}$$

The computation of the density of particles created is straightforward from (44) and the normalization condition²⁹ of the wave function

$$|\alpha|^2 + |\beta|^2 = 1 , \tag{46}$$

then

$$n = |\beta|^2 = \left[\left(\frac{|\beta|^2}{|\alpha|^2} \right)^{-1} + 1 \right]^{-1} . \tag{47}$$

It is worth mentioning that, thanks to the relations (43) and (46) we did not have to compute the normalization constants \mathfrak{C}_∞^+ , \mathfrak{C}_∞^- and \mathfrak{C}_0^+ .

4. Results and Discussion

Let us analyze the asymptotic behavior of (47) when the electric field vanishes. Taking into account that $\sinh(z) \sim \Theta(z)e^{|z|}/2$ as $z \rightarrow \infty$, we readily obtain

$$n \approx \frac{|\beta|^2}{|\alpha|^2} = \exp(-2\pi k_\perp), \quad (48)$$

which is the result obtained in Ref. 16. Expression (48) corresponds to a two-dimensional Fermi–Dirac thermal distribution. In the case of strong electric fields the density number of scalar particles created takes the form

$$n \approx e^{-\pi k_\perp - \frac{\pi m^2}{2eE}}, \quad (49)$$

showing that the density of particles created by the cosmological background and the electric field (49) is a Fermi distribution with a chemical potential proportional to $\frac{m^2}{eE}$. Integrating the particle density n (49) on momentum we obtain the total number of particles created per unit volume.

$$N = \frac{1}{V} \int n dk_x dk_y dk_z = \frac{1}{t^2(2\pi)^2} \int nk_\perp dk_\perp dk_z. \quad (50)$$

In order to carry out the integration we have to notice that n does not depend on k_z and consequently integration over k_z is equivalent to the substitution^{6,19} $\int dk_z \rightarrow eET$, where T is the time of interaction of the external field. Substituting (49) into (50), we obtain that the total number N of particles per unit volume takes the form

$$N \approx \frac{eE}{4\pi^4 T} e^{-\frac{\pi m^2}{2eE}}. \quad (51)$$

Result (51) resembles the number of particles created by a constant electric field in a Minkowski space.^{6,19} It is worth mentioning that the number N of particles per unit volume is inversely proportional to T^{-1} and vanishes as $T \rightarrow \infty$. The volume expansion of the anisotropic universe (1) is faster than the particle creation process, therefore N becomes negligible for large values of T . Since the gravitational density $\rho = 1/(8\pi t^2)$, decreases faster than (51), the particle creation mechanism effectively isotropizes in the anisotropic universe (1) in the presence of strong electric fields.

The results (49) and (51) show that the anisotropic cosmological background (1), as well the constant electric field, contribute to the creation of scalar particles. The quasiclassical method gives a recipe for obtaining the positive and negative frequency modes even when space–time is not static and an external source is present. The presence of the anisotropy with a constant electric field gives rise to a particle distribution that is thermal only in the asymptotic field regime. The method and results presented in this paper could be of help to discuss quantum effects in more realistic anisotropic cosmological scenarios.

Acknowledgments

The authors wish to thank A. Mishra for valuable comments. One of the authors (V.M.V.) acknowledges a fellowship from the Alexander von Humboldt Stiftung.

References

1. L. Parker, *Phys. Rev. Lett.* **21**, 562 (1968).
2. L. Parker, *Asymptotic Structure of Space-Time* (Plenum, 1977), p. 107.
3. Ya. Zeldovich and A. A. Starobinskii, *Sov. Phys. JETP* **34**, 1159 (1971).
4. S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge Univ. Press, 1991).
5. D. M. Chitre and J. B. Hartle, *Phys. Rev.* **D16**, 251 (1977).
6. A. A. Grib, S. G. Mamaev and V. M. Mostepanenko, *Quantum Vacuum Effects in Strong Fields* (Energoatomizdat, 1988).
7. I. L. Bukhbinder, *Izv. Vuzov. Fiz.* **7**, 3 (1980).
8. S. Gavrilov, D. M. Gitman and S. D. Odintsov, *Int. J. Mod. Phys.* **A12**, 4837 (1997).
9. N. D. Birrel and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge Univ. Press, 1982).
10. H. Nariai, *Prog. Theor. Phys.* **59**, 1532 (1978).
11. M. V. Fischetti, J. B. Hartle and B. L. Hu, *Phys. Rev.* **D20**, 1757 (1979).
12. V. Sahni, *Class. Quantum Grav.* **1**, 579 (1984).
13. I. L. Bukhbinder and D. M. Gitman, *Izv. Vuzov Fiz.* **3**, 90 (1979).
14. I. L. Bukhbinder, *Izv. Vuzov Fiz.* **4**, 55 (1979).
15. I. L. Bukhbinder and S. D. Odintsov, *Izv. Vuzov. Fiz.* **5**, 93 (1983).
16. V. M. Villalba, *Int. J. Theor. Phys.* **36**, 1321 (1997).
17. V. M. Villalba, *Phys. Rev.* **D52**, 3742 (1995).
18. J. Schwinger, *Phys. Rev.* **82**, 664 (1951).
19. A. I. Nikishov, *Trudy FIAN No. 111* (Nauka, 1979).
20. J. Garriga, *Phys. Rev.* **D49**, 6343 (1994).
21. I. L. Bukhbinder and S. D. Odintsov, *Izv. Vuzov. Fiz.* **5**, 12 (1982).
22. K. H. Lotze, *Class. Quantum Grav.* **2**, 351 (1985).
23. K. H. Lotze, *Class. Quantum Grav.* **2**, 363 (1985).
24. V. M. Villalba *Phys. Rev.* **D60**, 127501 (1999).
25. G. V. Shishkin and V. M. Villalba, *J. Math. Phys.* **30**, 2132 (1989).
26. V. M. Villalba *Mod. Phys. Lett.* **A8**, 2351 (1993).
27. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, 1974).
28. N. N. Lebedev, *Special Functions and Their Applications* (Dover, 1972).
29. T. Mishima and A. Nakayama, *Phys. Rev.* **D37**, 348 (1988).