Ableitung der Formel aus Kapitel 1 für die Gaussische Krümmung

Formel aus Weinberg (Einleitung)

1.7:

$$K(u_1,u_2) = \frac{1}{2g}\left[2\frac{\partial^2 g_{12}}{\partial u_1 \partial u_2} - \frac{\partial^2 g_{11}}{\partial u_2^2} - \frac{\partial^2 g_{22}}{\partial u_1^2}\right]$$

$$-\frac{g_{22}}{4g^2}\left[\frac{\partial g_{11}}{\partial u_1}\left(2\frac{\partial g_{12}}{\partial u_2} - \frac{\partial g_{22}}{\partial u_1}\right) - \left(\frac{\partial g_{11}}{\partial u_2}\right)^2\right]$$

$$+\frac{g_{12}}{4g^2}\left[\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{22}}{\partial u_2} - 2\frac{\partial g_{11}}{\partial u_2}\frac{\partial g_{22}}{\partial u_1} + \left(2\frac{\partial g_{12}}{\partial u_1} - \frac{\partial g_{11}}{\partial u_2}\right)\left(2\frac{\partial g_{12}}{\partial u_2} - \frac{\partial g_{22}}{\partial u_1}\right)\right]$$

$$-\frac{g_{11}}{4g^2}\left[\frac{\partial g_{22}}{\partial u_2}\left(2\frac{\partial g_{12}}{\partial u_1} - \frac{\partial g_{11}}{\partial u_2}\right) - \left(\frac{\partial g_{22}}{\partial u_1}\right)^2\right] \overset{?}{=} -\frac{R}{2}$$

Riemann Krümmungsskalar

Es ist in zwei Dimensionen

$$g_{\mu\nu} = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}, \quad g = \det g_{\mu\nu} = g_{11}g_{22} - (g_{12})^2$$

inverse Matrix

$$g^{\mu\nu} = \begin{pmatrix} g^{11} & g^{12} \\ g^{12} & g^{22} \end{pmatrix} = \begin{pmatrix} g_{22}/g & -g_{12}/g \\ -g_{12}/g & g_{11}/g \end{pmatrix} \qquad A^{-1} = \frac{1}{\det A}\,\mathrm{adj}(A)$$

Adjunkte

Laut S. 9 Kap 5 ist

$$R = g^{\beta\nu}R_{\beta\nu} = 2\left(g^{11}g^{22} - (g^{12})^2\right)R_{1212} = \frac{2}{g^2}\left(g_{22}g_{11} - (g_{12})^2\right)R_{1212}$$

$$= \frac{2}{g}R_{1212}$$

$$\leadsto K \overset{?!}{=} -\frac{R_{1212}}{g}$$

Nun ist

$$R_{\mu\nu\alpha\beta} = \frac{1}{2}\left(\frac{\partial^2 g_{\alpha\mu}}{\partial x^\beta \partial x^\nu} - \frac{\partial^2 g_{\beta\mu}}{\partial x^\alpha \partial x^\nu} - \frac{\partial^2 g_{\alpha\nu}}{\partial x^\beta \partial x^\mu} + \frac{\partial^2 g_{\beta\nu}}{\partial x^\mu \partial x^\nu}\right)$$

$$+ g_{\rho\lambda}\left(\Gamma^\rho_{\alpha\mu}\Gamma^\lambda_{\beta\nu} - \Gamma^\rho_{\beta\mu}\Gamma^\lambda_{\alpha\nu}\right)$$

$$\Rightarrow \quad R_{1212} = \frac{1}{2}\left(\frac{\partial^2 g_{11}}{\partial u_2 \partial u_2} - \frac{\partial g_{21}}{\partial u_1 \partial u_2} - \frac{\partial^2 g_{12}}{\partial u_2 \partial u_1} + \frac{\partial^2 g_{22}}{\partial u_1 \partial u_1}\right)$$

$$+ g_{\rho\lambda}\left(\Gamma^\rho_{11}\Gamma^\lambda_{22} - \Gamma^\rho_{21}\Gamma^\lambda_{12}\right)$$

und

$$\Gamma^\rho_{\alpha\mu} = \frac{1}{2} g^{\nu\rho}\left(\frac{\partial g_{\mu\nu}}{\partial x^\alpha} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\alpha}}{\partial x^\nu}\right)$$

Somit als wichtige Vereinfachung

$$g_{\rho\lambda}\Gamma^\rho_{\alpha\mu} = g_{\rho\lambda}\, g^{\nu\rho}\, \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\alpha} + \frac{\partial g_{\alpha\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\alpha}}{\partial x^\nu}\right)$$

$$= \frac{1}{2} g_{\lambda\rho}\, g^{\rho\nu}\left(\text{"} \right) = \frac{1}{2}\,\delta^\nu_\lambda\left(\quad \right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{\mu\lambda}}{\partial x^\alpha} + \frac{\partial g_{\alpha\lambda}}{\partial x^\mu} - \frac{\partial g_{\mu\alpha}}{\partial x^\lambda}\right)$$

Damit

Damit

$$g_{\varrho\lambda}\,\Gamma^{\varrho}_{11} = \frac{1}{2}\left(2\frac{\partial g_{1\lambda}}{\partial u_1} - \frac{\partial g_{11}}{\partial u_\lambda}\right)$$

$$g_{\varrho\lambda}\,\Gamma^{\varrho}_{12} = \frac{1}{2}\left(\frac{\partial g_{2\lambda}}{\partial u_1} + \frac{\partial g_{1\lambda}}{\partial u_2} - \frac{\partial g_{12}}{\partial u_\lambda}\right)$$

und es wird

$$\left.R_{1212} = \frac{1}{2}\left(\frac{\partial^2 g_{11}}{\partial u_2 \partial u_2} + \frac{\partial^2 g_{22}}{\partial u_1 \partial u_1} - 2\frac{\partial^2 g_{12}}{\partial u_1 \partial u_2}\right)\right\}$$

Anmerkung: der Term passt schon zu $(-)K \cdot g$

$$+ \frac{1}{2}\left(2\frac{\partial g_{11}}{\partial u_1}\right)\Gamma^{1}_{22}$$

$$+ \frac{1}{2}\left(2\frac{\partial g_{12}}{\partial u_1} - \frac{\partial g_{11}}{\partial u_2}\right)\Gamma^{2}_{22}$$

$$- \frac{1}{2}\left(\frac{\partial g_{11}}{\partial u_2}\right)\Gamma^{1}_{12}$$

$$- \frac{1}{2}\left(\frac{\partial g_{22}}{\partial u_1}\right)\Gamma^{2}_{12}$$

Berechnung der vier Christoffel-Symbole

$$\Gamma^1_{22} = \frac{1}{2} g^{\nu 1}\left(\frac{\partial g_{2\nu}}{\partial u_2} + \frac{\partial g_{2\nu}}{\partial u_2} - \frac{\partial g_{22}}{\partial u_\nu}\right) =$$

$$= \frac{1}{2} g^{11}\left(2\frac{\partial g_{12}}{\partial u_2} - \frac{\partial g_{22}}{\partial u_1}\right) + \frac{1}{2} g^{21}\left(2\frac{\partial g_{22}}{\partial u_2} - \frac{\partial g_{22}}{\partial u_2}\right)$$

$$= \frac{1}{2g} g_{22}\left(2\frac{\partial g_{12}}{\partial u_2} - \frac{\partial g_{22}}{\partial u_1}\right) + \frac{1}{2g} g_{12}\left(-\frac{\partial g_{22}}{\partial u_2}\right)$$

$$\Gamma^2_{22} = \frac{1}{2} g^{\nu 2}\left(\frac{\partial g_{2\nu}}{\partial u_2} + \frac{\partial g_{2\nu}}{\partial u_2} - \frac{\partial g_{22}}{\partial u^\nu}\right)$$

$$= \frac{1}{2} g^{12}\left(2\frac{\partial g_{12}}{\partial u_2} - \frac{\partial g_{22}}{\partial u_1}\right) + \frac{1}{2} g^{22}\frac{\partial g_{22}}{\partial u_2}$$

$$= \frac{1}{2g} g_{12}\left(\frac{\partial g_{22}}{\partial u_1} - 2\frac{\partial g_{12}}{\partial u_2}\right) + \frac{1}{2g} g_{11}\frac{\partial g_{22}}{\partial u_2}$$

$$\Gamma^1_{12} = \frac{1}{2} g^{1\nu}\left(\frac{\partial g_{2\nu}}{\partial u_1} + \frac{\partial g_{1\nu}}{\partial u_2} - \frac{\partial g_{12}}{\partial u_\nu}\right)$$

$$= \frac{1}{2} g^{11}\left(\frac{\partial g_{11}}{\partial u_2}\right) + \frac{1}{2} g^{12}\left(\frac{\partial g_{22}}{\partial u_1}\right) = \frac{1}{2g} g_{22}\frac{\partial g_{11}}{\partial u_2} + \frac{1}{2g} g_{12}\left(-\frac{\partial g_{22}}{\partial u_1}\right.$$

$$\Gamma^2_{12} = \frac{1}{2} g^{2\nu}\left(\frac{\partial g_{2\nu}}{\partial u_1} + \frac{\partial g_{1\nu}}{\partial u_2} - \frac{\partial g_{12}}{\partial u_\nu}\right)$$

$$= \frac{1}{2} g^{21}\frac{\partial g_{12}}{\partial u_2} + \frac{1}{2} g^{22}\frac{\partial g_{22}}{\partial u_1} = \frac{1}{2g} g_{12}\left(-\frac{\partial g_{11}}{\partial u_2}\right) + \frac{1}{2g} g_{11}\frac{\partial g_{22}}{\partial u_1}$$

Wir sortieren nun nach Termen g_{11}, g_{22} und g_{12} der letzten vier Zeilen des Ausdruckes für R_{1212} auf S. (A3) (A1)

g_{11}: Γ^2_{22} und Γ^2_{12} haben einen Term mit g_{11}

$$\frac{1}{4g^2}\left(2\frac{\partial g_{12}}{\partial u_1}-\frac{\partial g_{11}}{\partial u_2}\right)\cdot\frac{\partial g_{22}}{\partial u_2}-\frac{1}{4g^2}\left(\frac{\partial g_{22}}{\partial u_1}\right)^2$$

Dieses stimmt bis auf den Faktor $(-1)\frac{1}{g}$ mit dem g_{11}-Term von $K(u_1,u_2)$ auf S. (A1) "überein", wie schon der erste Term des R_{1212} mit den zweiten Ableitungen der Metrik nach den Koordinaten.

g_{22}: Γ^1_{22} und Γ^1_{12} tragen bei

$$\frac{1}{4g^2}\left(\frac{\partial g_{11}}{\partial u_1}\right)\left(2\frac{\partial g_{12}}{\partial u_2}-\frac{\partial g_{22}}{\partial u_1}\right)-\frac{1}{4g^2}\left(\frac{\partial g_{11}}{\partial u_2}\right)^2$$

auch wieder $(-1)\cdot\frac{1}{g}$ "Übereinstimmung"

g_{12}: g_{12} taucht überall auf

$$\left(-\frac{1}{4g^2}\right)\left(\frac{\partial g_{11}}{\partial u_1}\frac{\partial g_{22}}{\partial u_2}\right)+\left(\frac{1}{4g^2}\right)\left(2\frac{\partial g_{12}}{\partial u_1}-\frac{\partial g_{11}}{\partial u_2}\right)\left(\frac{\partial g_{22}}{\partial u_1}-2\frac{\partial g_{12}}{\partial u_2}\right)$$

$$\left(+\frac{1}{4g^2}\right)\left(\frac{\partial g_{11}}{\partial u_2}\cdot\frac{\partial g_{22}}{\partial u_1}\right)+\left(\frac{1}{4g^2}\right)\frac{\partial g_{22}}{\partial u_1}\frac{\partial g_{11}}{\partial u_2}$$

auch hier wieder $(-1)\cdot\frac{1}{g}$ "Übereinstimmung".

Damit ist die Formel auf Seite (A1) u.d.T. gezeigt.