Tutorial “General Relativity”

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Lecturer: Prof. Dr. C. Greiner
Tutor: Hendrik van Hees

Sheet No. 5 – Solutions

will be discussed on Jan/17/17

1. Barometric formula in General Relativity

In the lecture we have shown that an ideal fluid obeys the equation of motion,

$$T^{\alpha\beta}{}_{\|\beta} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\beta} \left[\sqrt{|g|}(\epsilon + P) u^\alpha u^\beta\right] + (\epsilon + P)\Gamma^\alpha_{\beta\nu} u^\beta u^\nu - g^{\alpha\beta} \frac{\partial P}{\partial x^\beta} = 0. \tag{1}$$

Now consider an ideal fluid in a stationary gravitational field in hydrostatic ($u^j = 0$ for $j \in \{1, 2, 3\}$) equilibrium. This case is relevant for the description of neutron stars, which have a strong gravitational field and thus have to be treated within the General Theory of Relativity.

(a) Show that $u^0 = \frac{1}{\sqrt{g_{00}}}$.

Solution: Since $u_\mu u^\mu = g_{\mu\nu} u^\mu u^\nu = g_{00}(u^0)^2 = 1$ we immediately find $u^0 = 1/\sqrt{g_{00}}$.

(b) Explain, why the first term in (1) is identically zero.

Solution: The expression is summed over β. The term for $\beta = 0$ vanishes since by assumption both the metric coefficients, the energy density, pressure, and the four-velocity are time independent. For $\beta \in \{1, 2, 3\}$ the expression vanishes, because then $u^\beta = 0$.

(c) Prove that the equation simplifies to a generalized barometric formula,

$$\frac{\partial P}{\partial x^\beta} = -\frac{1}{2}(\epsilon + P) \frac{\partial g_{00}}{\partial x^\beta} \frac{1}{g_{00}} = 0, \tag{2}$$

i.e., for the spatial components,

$$\partial_j P = -\frac{\epsilon + P}{2} \partial_j (\ln g_{00}) \quad \text{for} \quad j \in \{1, 2, 3\}. \tag{3}$$

Solution: For the remaining term of (1) we have

$$\Gamma^\alpha{}_{\mu\nu} u^\mu u^\nu = \Gamma^\alpha{}_{00}(u^0)^2 \stackrel{\text{(a)}}{=} \frac{1}{g_{00}} \Gamma^\alpha{}_{00}. \tag{4}$$

Further the Christoffel symbols are defined by

$$\Gamma_{\gamma\mu\nu} := g_{\gamma\alpha} \Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}(\partial_\mu g_{\gamma\nu} + \partial_\nu g_{\gamma\mu} - \partial_\gamma g_{\mu\nu}). \tag{5}$$

For $\mu = \nu = 0$ the first two expressions in the bracket vanish, because $\partial_0 g_{\mu\nu} = 0$ and thus

$$\Gamma_{\gamma 00} = -\frac{1}{2} \partial_\gamma g_{00}. \tag{6}$$

So contracting (1) with $g_{\gamma\alpha}$, we find

$$\partial_\gamma P = (\epsilon + P)\Gamma_{\gamma 00} \frac{1}{g_{00}} \stackrel{(6)}{=} -\frac{1}{2g_{00}} \partial_\gamma g_{00}(\epsilon + P) = -\frac{(\epsilon + P)}{2} \partial_\gamma \ln(g_{00}). \tag{7}$$

(d) Show that in the non-relativistic limit

$$\vec{\nabla} P = -\varrho \vec{\nabla} \phi_{\text{grav}}, \tag{8}$$

where ϕ_{grav} denotes the Newtonian gravitational potential.

Solution: In the non-relativistic limit we have

$$g_{00} \simeq 1 + \frac{2\phi_{\text{grav}}}{c^2} \Rightarrow \ln(g_{00}) \simeq 2\frac{\phi_{\text{grav}}}{c^2}. \tag{9}$$

Since also $\epsilon \simeq \rho c^2 \gg P$, where ρ is the mass density, we finally get the usual barometric formula (8).

Note: To solve (1) or (8) one needs an equation of state and the assumption that the change of the state due to the motion of the fluid is adiabatic (which is implied by the assumption of ideal fluid dynamics, which is already used in the ansatz of the energy-momentum tensor in (1). A typical equation of state is the polytrope,

$$P = C\rho^n, \quad C, n = \text{const.} \tag{10}$$

E.g., for an ideal gas of diatomic molecules (like N_2 or O_2 in the Earth's atmosphere) one has $n = 1.4$.

Close to the Earth we have in a Cartesian coordinate system (x, y, z)

$$\phi_{\text{grav}} = gz, \tag{11}$$

and $P = P(z)$. Using (10) then (8) leads to the ordinary differential equation,

$$P' = -\left(\frac{P}{C}\right)^{1/n} g, \tag{12}$$

i.e.,

$$gz = -\int_{P_0}^{P} \mathrm{d}\tilde{P} \left(\frac{C}{\tilde{P}}\right)^{1/n} = \frac{C^{1/n} n}{n-1} \left(P_0^{1-1/n} - P^{1-1/n}\right). \tag{13}$$

With $C = P_0/\rho_0^n$ we finally get

$$P = P_0 \left(1 - \frac{n-1}{n} \frac{\rho_0}{P_0} gz\right)^{n/(n-1)}. \tag{14}$$

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2. Geodesics equation from ideal hydrodynamics

In this problem we want the show that the equations of motion for free fall leads to the equation for (timelike) geodesics via hydrodynamics of an ideal fluid. The generally covariant equations of motion of an ideal fluid is given by the local conservation of energy and momentum, i.e.,

$$T^{\mu\nu}{}_{\|\nu} = 0 \tag{15}$$

for the energy-momentum tensor

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu - P g^{\mu\nu}, \tag{16}$$

with the energy density ϵ, pressure P, and four-velocity field u^μ normalized such that $u_\mu u^\mu = 1$.

(a) Show that

$$u^{\mu}{}_{\|\nu} u_{\mu} = 0. \tag{17}$$

Solution: Since $u^{\mu}u_{\mu} = 1$ and due to the product rule for partial derivativs we find

$$(u^{\mu}u_{\mu})_{\|\nu} = \partial_{\nu}(u^{\mu}u_{\mu}) = 0 = u^{\mu}{}_{\|\nu}u_{\mu} + u^{\mu}u_{\mu\|\nu} = 2u^{\mu}{}_{\|\nu}u_{\mu} \;\Rightarrow\; u^{\mu}{}_{\|\nu}u_{\mu} = 0. \tag{18}$$

In the first step we have used that the covariant derivative of a scalar is just the partial derivative, and in the last step we have used the Ricci theorem according to which all covariant derivatives of the metric vanish, i.e.,

$$u^{\mu}u_{\mu\|\nu} = u^{\mu}(u^{\rho}g_{\rho\mu})|_{\|\nu} = u^{\mu}(u^{\rho}{}_{\|\nu}g_{\rho\mu} + u^{\rho}g_{\rho\mu\|\nu}) = u^{\mu}g_{\rho\mu}u^{\rho}{}_{\|\nu} = u_{\rho}u^{\rho}{}_{\|\nu}. \tag{19}$$

(b) Show from (15) and the contraction of the equation with u_{μ} that

$$(\epsilon + P)u^{\mu}{}_{\|\nu}u^{\nu} = (g^{\mu\nu} - u^{\mu}u^{\nu})P_{\|\nu}\,. \tag{20}$$

Solution: Since the product rule holds for covariant derivatives, we have

$$\begin{aligned} T^{\mu\nu}{}_{\|\nu} &= (\epsilon + P)_{\|\nu}u^{\mu}u^{\nu} + (\epsilon + P)u^{\mu}u^{\nu}{}_{\|\nu} + (\epsilon + P)u^{\mu}{}_{\|\nu}u^{\nu} - P_{\|\nu}g^{\mu\nu} - Pg^{\mu\nu}{}_{\|\nu} \\ &= u^{\mu}[(\epsilon + P)u^{\nu}]_{\|\nu} + (\epsilon + P)u^{\mu}{}_{\|\nu}u^{\nu} - P_{\|\nu}g^{\mu\nu} = 0, \end{aligned} \tag{21}$$

where we have again used Ricci's theorem.
Contracting with u_{μ} and using $u_{\mu}u^{\mu} = 1$ and (17) gives

$$[(\epsilon + P)u^{\nu}]_{\|\nu} - u^{\nu}P_{\|\nu} = 0. \tag{22}$$

Substitution of this in the first term of (21) gives

$$-(g^{\mu\nu} - u^{\mu}u^{\nu})P_{\|\nu} + (\epsilon + P)u^{\mu}{}_{\|\nu}u^{\nu} = 0, \tag{23}$$

which is (20).

(c) For "dust", i.e., a fluid consisting of non-interacting particles, which means that $P = 0$, this obviously implies that

$$u^{\mu}{}_{\|\nu}u^{\nu} = 0. \tag{24}$$

Show that this implies that the flow lines, i.e., the trajectories of the dust particles, are geodesics.
Hint: The flow lines are defined by the differential equation

$$\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s} = u^{\mu}. \tag{25}$$

Solution: We write out the covariant derivatives in (24)

$$u^{\nu}(\partial_{\nu}u^{\mu} + \Gamma^{\mu}{}_{\nu\rho}u^{\rho}) = 0. \tag{26}$$

Now substituting $u^{\nu} = \mathrm{d}x^{\nu}/\mathrm{d}s$ in the first term gives with the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}s}u^{\mu} + \Gamma^{\mu}{}_{\nu\rho}u^{\nu}u^{\rho} = 0. \tag{27}$$

Subsituting $u^{\mu} = \mathrm{d}x^{\mu}\mathrm{d}/s$ everywhere finally yields the equations for geodesics,

$$\frac{\mathrm{d}^2x^{\mu}}{\mathrm{d}s^2} + \Gamma^{\mu}{}_{\nu\rho}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}\frac{\mathrm{d}x^{\rho}}{\mathrm{d}s} = 0. \tag{28}$$

Note: The results show that energy-momentum conservation for freely falling particles implies that they follow (timelike) geodesics of spacetime, as is also implied by the (weak) equivalence principle used in the lecture to motivate the description of gravity as the curvature of a non-Euclidean spacetime manifold.

According to (20) the flow lines of a free falling ideal fluid are no geodesics, but on the fluid particles an additional force is acting due to its pressure.