# Tutorial "General Relativity"

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## Sheet No. 6 – Solutions

will be discussed on January 31, 2017

**1. Newtonian limit**

Find the relation between the geodesic equation and the Newtonian equation of motion for a particle moving in a static gravitational field, i.e. show that Newtonian gravity can be described by a metric of the form

$$\mathrm{d}s^2 = \left[1 + \frac{2}{c^2}\phi(\vec{x})\right] c^2 \mathrm{d}t^2 - g_{jk}\mathrm{d}x^j \mathrm{d}x^k \tag{1}$$

where $\phi(\vec{x}) = -\frac{GM}{r}$ is the gravitational potential.

**Hint:** Use the ansatz $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(\vec{x})$ with $|h_{\mu\nu}| \ll 1$ for a weak static gravitational field in the equations of motion for a freely falling particle and use the non-relativistic approximation of the equations of motion,

$$\frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = -\vec{\nabla}\phi(\vec{x}) \tag{2}$$

to find the relation between $h_{00}$ and $\phi$.

**Remark:** Since in a static gravitational field $\mathrm{d}s^2$ should be time independent, it cannot change under time reversal, $t \to -t$, and thus $g_{0j}$ ($j \in \{1,2,3\} = 0$). Note that in this way, we cannot make any further statement about the spatial components of the metric, $g_{jk}$, in the non-relativistic limit.

**Solution:** We can formally take $GM$ as the small expansion parameter. From the non-relativistic equation of motion we read off that $\ddot{\vec{x}} \propto GM$, and this is also the parametric dependence of $\dot{\vec{x}}$. The equation of motion of the freely falling particle is the geodesics equation, which is most easily found by the least-action principle with the Lagrangian

$$L = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu, \tag{3}$$

where the world-line parameter can be chosen as the proper time of the particle, so that

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = c^2. \tag{4}$$

Since the velocities are small, we have $\mathrm{d}\tau \simeq \mathrm{d}t$ up to 2nd-order corrections in $GM$.

Now we look at the equations of motion, using the Euler-Lagrange equations. The canonical momentum reads

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = g_{\mu\nu}\dot{x}^\nu, \tag{5}$$

and thus the equations of motion

$$\dot{p}_\mu = g_{\mu\nu}\ddot{x}^\nu + \partial_\rho g_{\mu\nu}\dot{x}^\rho \dot{x}^\nu \stackrel{!}{=} \frac{\partial L}{\partial x^\mu} = \frac{1}{2}\partial_\mu g_{\rho\nu}\dot{x}^\rho \dot{x}^\nu. \tag{6}$$

Raising the index leads finally to

$$\ddot{x}^\sigma = \frac{1}{2} g^{\sigma\mu}\partial_\mu g_{\rho\nu}\dot{x}^\rho \dot{x}^\nu - g^{\sigma\mu}\partial_\rho g_{\mu\nu}\dot{x}^\rho \dot{x}^\nu. \tag{7}$$

Since $\partial_0 g_{\mu\nu} = 0$ and for the spatial components $\dot{x}^\rho = \mathcal{O}(GM)$ and $\partial_r g_{\rho\nu} = \mathcal{O}(GM)$ the 2nd term on the right-hand side of Eq. (7) can be neglected, being of order $\mathcal{O}[(GM)^2]$. For the same reason in the 1st term on the right-hand side in the sum over the indices $\rho$ and $\nu$ only the term with $\rho = \nu = 0$ is of order $\mathcal{O}(GM)$, i.e., up to corrections of 2nd order in $GM$ we have

$$\ddot{x}^\sigma = \frac{1}{2} g^{\sigma\mu} \partial_\mu g_{00} \dot{x}^0 \dot{x}^0. \tag{8}$$

Now since we can take the derivatives wrt. to $\tau$ in (8) to be derivatives wrt. to $t$ to the same order in $GM$, i.e., we can set $\dot{x}^0 = c$ and set $g^{\sigma\mu} \simeq \eta^{\sigma\mu}$. Thus we get

$$\ddot{x}^\sigma = \frac{c^2}{2} \eta^{\sigma\mu} \partial_\mu g_{00}. \tag{9}$$

Now indeed for $\sigma = 0$, we get

$$\ddot{x}^0 = 0, \tag{10}$$

because $\partial_0 g_{00} = 0$, and this is compatible with the power counting in $GM$, argued about above, i.e., we have $x^0 \simeq ct + \text{const}$. For the spatial components we finally get

$$\ddot{x}^s = -\frac{c^2}{2} \delta^{sm} \partial_m g_{00} \stackrel{!}{=} -\delta^{sm} \partial_m \phi, \tag{11}$$

and this yields

$$g_{00} \simeq \frac{2}{c^2} \phi + \text{const}. \tag{12}$$

If no matter is present, the spacetime must be described by the flat Minkowski spacetime, and thus we have const $= 1$. So we finally have, in the non-relativistic limit

$$g_{00} \simeq 1 + \frac{2}{c^2} \phi, \tag{13}$$

and this we had to show.

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*2.* **Schwarzschild solution with cosmological term**

The cosmological constant has been added by Einstein as a fudge factor. At the time it was believed that the universe was static. The Einstein equations however predict a dynamical universe. When the observations of Hubble proved beyond reasonable doubt that the universe was expanding, Einstein threw out the cosmological constant and claimed it to be *"Die größte Eselei meines Lebens"*.

Recent observations seem to indicate that some sort of "vacuum energy" is at work, so that the cosmological constant is coming back to style. The vacuum Einstein equations with a cosmological constant read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{14}$$

What is the influence (consequence) of $\Lambda$ on the Schwarzschild solution outside the star? To this purpose calculate the *modified* line element using the same ansatz as used in the lecture to derive the Schwarzschild metric given as

$$\mathrm{d}s^2 = c^2 \mathrm{d}t^2 \exp\nu - \mathrm{d}r^2 \exp\lambda - r^2 (\mathrm{d}\vartheta^2 + \mathrm{d}\varphi^2 \sin^2\theta) \tag{15}$$

Is the asymptotic limit for $r \to \infty$ still a Minkowski space-time?

**Solution:** We can use the Einstein tensor from the lecture to derive the equations of motion for the gravitational field. As in the case $\Lambda = 0$, we start with solving the equations for the 00- and 11-component:

$$G_{00} + \Lambda g_{00} = \exp\nu \left\{ \exp(-\lambda) \left[ \frac{1}{r^2} - \frac{\lambda'}{r} \right] - \frac{1}{r^2} + \Lambda \right\} \stackrel{!}{=} 0, \tag{16}$$

$$G_{11} + \Lambda g_{22} = -\frac{\nu'}{r} - \frac{1}{r^2} + \frac{\exp\lambda}{r^2} - \Lambda \exp\lambda = 0. \tag{17}$$

In (16) the expression within the curly bracket has to vanish. Since this is a linear differential equation of motion for $\lambda$ alone, it is easy to solve. It can be rearranged to

$$\frac{\mathrm{d}}{\mathrm{d}r}[r\exp(-\lambda)] = 1 - \Lambda r^2, \tag{18}$$

with the solution

$$\exp(-\lambda) = 1 - \frac{r_{\mathrm{S}}}{r} - \frac{\Lambda r^2}{3}. \tag{19}$$

The integration constant is the same Schwarzschild radius as in the spacetime without cosmological constant since close to the center we should get this limit, $r_{\mathrm{S}} = 2GM/c^2$.

To find also the solution for $\nu$ we can work with (16) and (17) as in the case for $\Lambda = 0$: Multiplying (16) by $\exp\lambda$ and adding it to (17) yields again

$$\nu' = -\lambda' \;\Rightarrow\; \nu = -\lambda + \nu_0. \tag{20}$$

Here, the integration constant has just the meaning of a constant factor in the 00-component of the metric tensor, which can be eliminated by rescaling the time coordinate by this factor. So we can set $\nu_0 = 0$ without loss of generality. Finally we arrive at the solution for the metric, which we note in terms of the invariant length element,

$$\mathrm{d}s^2 = \left(1 - \frac{r_{\mathrm{S}}}{r} - \frac{\Lambda r^2}{3}\right) c^2 \mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - \frac{r_{\mathrm{S}}}{r} - \frac{\Lambda r^2}{3}} - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2). \tag{21}$$

**Remarks:** As we see, for $r \to \infty$ the spacetime does not become asymptotically flat in this case. This is understandable, because the term with the cosmological constant can be interpreted as a special contribution to the energy-momentum tensor. In modern cosmology thus one refers to the cosmological constant also as "dark energy".

This issue is one of the greatest unsovled mysteries of contemporary physics. On the one hand the cosmological constant can be interpreted as the value of the energy density of the quantum-field theoretical vacuum of the Standard Model of elementary particle physics. On the other hand this model contains the Higgs field as a scalar boson, which implies a quadratically divergent mass renormalization term for the Higgs-particle mass. This term, when extrapolated to the huge energy scale given by the "Planck mass", leads to a huge cosmological constant.

On the other hand, accurate measurements of the fluctuations in the cosmic microwave background and the relation between redshift and distance of far distant type-I supernovae ("Hubble diagram") leads to a cosmological constant that is by a factor of about $10^{122}$ smaller than expected from the estimate from the interpretation as the vacuum energy density of the Standard Model!

Further, the cosmological constant in our universe is so small that there is no hint from the motion of the planets around the Sun in our solar system and not even for the relative motion of galaxies within galaxy clusters.

As we see from (21) the Schwarzschild solution with positive cosmological constant implies that there is not only the horizon given by the Schwarzschild radius $r < r_{\mathrm{s}}$ but also on a much larger scale $r_{\mathrm{c}} \gg r_{\mathrm{s}}$. Indeed of $r \gg r_{\mathrm{S}}$ the metric becomes singular for $1 - \Lambda r_{\mathrm{c}}^2/3 \simeq 0$, i.e., at $r_{\mathrm{c}} \simeq \sqrt{3/\Lambda}$. This should be the order of the distance from the center, where the influence of the cosmological constant should become visible by the motion of bodies around some center of mass detributions. Since there's nothing like this visible within our solar system we can deduce that $r_{\mathrm{c}} \gg 10^{10}\,\mathrm{km}$. Since one even does not see any influence of $\Lambda$ on the relative motion of galaxies within galaxy clusters leads to $r_{\mathrm{c}} \gg 10^{23}\,\mathrm{km}$.