Tutorial “General Relativity”

Winter term 2016/2017

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Sheet No. 7

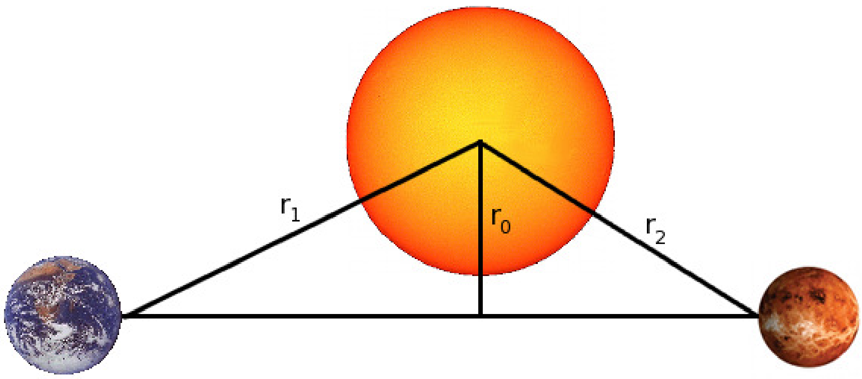
will be discussed on February 07, 2017

1. Shapiro Delay

Consider a light beam that moves from Earth to Venus and back, coming close to the Sun (see the figure). Non-relativistically the total “radar time” needed to come back to Earth obviously is

$$t_{\text{nrel}} = 2\left(\sqrt{r_1^2 - r_0^2} + \sqrt{r_2^2 - r_0^2}\right), \tag{1}$$

where r_0 is the distance of closest approach of the light beam to the Sun. The task of this problem is to evaluate the radar time taking into account the gravity of the Sun according to General Relativity.



Start from the Schwarzschild invariant-length element,

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right) c^2 \mathrm{d}t^2 - \frac{\mathrm{d}r^2}{(1 - \frac{2m}{r})} - r^2(\mathrm{d}\phi^2 + \sin^2\theta \mathrm{d}\phi^2), \quad 2m = r_S = \frac{2GM}{c^2}, \tag{2}$$

and use the result for the null geodesic from the lecture, describing the light beam (by an appropriate choice of the coordinate system according to the initial conditions running in the plane $\vartheta = \pi/2 = \text{const}$),

$$\left(1 - \frac{2m}{r}\right) c^2 \dot{t}^2 - \frac{\dot{r}^2}{(1 - \frac{2m}{r})} - \frac{h^2}{r^2} = 0. \tag{3}$$

The dot indicates the derivative with respect to an arbitrary affine parameter[1]. Here

$$h = r^2 \dot{\phi} = \text{const} \tag{4}$$

follows from the rotational symmetry formalized by the fact that ϕ is a cyclic variable of the Lagrangian, $L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu/2$ leading to the equation of the geodesic (see the note at the end of this document). The same holds true for t since the Schwarzschild metric is static, i.e.,

$$\left(1 - \frac{2m}{r}\right) \dot{t} = A = \text{const}. \tag{5}$$

[1]Note that for a light beam, which formally follows the geodesic of a massless particle, there does not exist a physical definition of such a parameter, like proper time for a massive particle!

(a) Express h in terms of r_0 and A, evaluate $\mathrm{d}r/\mathrm{d}t = \dot{r}/\dot{t}$ from (3) and (4).

(b) With that result show, that the radar time for the signal running forth and back from Venus is given by[2]

$$ct_{\text{radar}} = 2[F(r_1) + F(r_2)], \tag{6}$$

where

$$F(r_1) = \int_{r_0}^{r_1} \frac{\mathrm{d}r}{1 - \frac{2m}{r}} \left[1 - \frac{1 - 2m/r}{1 - 2m/r_0} \left(\frac{r_0}{r}\right)^2\right]^{-1/2}, \quad r_1 > r_0. \tag{7}$$

(c) Evaluate the integral (7) approximately by expanding up to linear order in m, assuming that $2m/r = r_S/r \ll 1$ along the entire worldline of the light beam.

Hint: The result is

$$F(r_1) = 2m \ln\left(\frac{r_1 + \sqrt{r_1^2 - r_0^2}}{r_0}\right) + m\sqrt{\frac{r_1 - r_0}{r_1 + r_0}} + \mathcal{O}\left(\frac{m^2}{r_0^2}\right). \tag{8}$$

[2]We neglect the motion of the planets during the very short travel time of the light beam.