Quasistationary electrodynamics – fundamentals

Hendrik van Hees

January 12, 2019

1 Quasistationary approximation in circuit theory

The understanding of the quasistationary approximation is most easily explaned by example of **AC-circuit theory** for circuits with simple elements like resistors, capacitors, and inductances. We shall also restrict ourselves to the electromagnetic field, charge and current distributions with **harmonic time dependence**. Here we follow the textbook [Reb07].

The basic idea of the approximation is to neglect the "**displacement current**", i.e.,

$$\vec{j}_{\mathrm{D}} = \partial_t \vec{D}. \tag{1}$$

To simplify the task we work with complex-valued fields, i.e., wer write for the electric field

$$\vec{E}(t,\vec{r}) = \mathrm{Re}[\vec{E}_c(\vec{r})\exp(-\mathrm{i}\omega t)] \tag{2}$$

and analogously for all other field quantities.

The simplification of the complex notation is that we can use them to solve problems since the **Maxwell equations** are linear equations with real coefficients, and thus we can first solve them using the complex fields and then take their real part, which represent the physical fields.

For fields with harmonic time dependence we can thus write

$$\vec{E}'(t,\vec{r}) = \vec{E}_c(\vec{r})\exp(-\mathrm{i}\omega t) \tag{3}$$

and analogously for all the other fields.

We also assume simple linear **constitutive equations**, i.e.,

$$\vec{D} = \epsilon \vec{E}, \quad \vec{B} = \mu \vec{H}. \tag{4}$$

Since with the above assumed harmonic time dependence each time derivative can be substituted by the rule, $\partial_t \rightarrow -\mathrm{i}\omega$, the Maxwell equations,

$$\mathrm{curl}\,\vec{E} + \partial_t \vec{B} = 0, \tag{5}$$

$$\mathrm{div}\,\vec{B} = 0, \tag{6}$$

$$\mathrm{curl}\,\vec{H} = \vec{j} + \partial_t \vec{D}, \tag{7}$$

$$\mathrm{div}\,\vec{D} = \rho, \tag{8}$$

simplify to

$$\operatorname{curl} \vec{E}_c - \mathrm{i}\omega \vec{B}_c = 0, \tag{9}$$

$$\operatorname{div} \vec{B}_c = 0, \tag{10}$$

$$\operatorname{curl} \vec{H}_c = \vec{j}_c - \mathrm{i}\omega \vec{D}_c, \tag{11}$$

$$\operatorname{div} \vec{D}_c = \rho. \tag{12}$$

As already mentioned, the quasistationary approximation mostly consists of neglecting the displacement-current densiti $\vec{j}_{\mathrm{D}} = \partial_t \vec{D}$ in (7) or the corresponding term $\vec{j}_{\mathrm{D}c} = -\mathrm{i}\omega \vec{D}_c$ in (11). Then for $\vec{H}_c$ and $\vec{B}_c = \mu \vec{H}_c$ the simpler rules of magnetostatics apply.

We also note that (10) is automatically fulfilled already due to (9):

$$\mathrm{i}\omega \operatorname{div} \vec{B}_c = \operatorname{div} \operatorname{curl} \vec{E}_c = 0. \tag{13}$$

Now we consider the most important application of the quasistationary approximation, AC-circuit theory, where we have quite compact building blocks like resistors, capacitors, and inductances connected with thin wires. Within the wires we assume the validity of Ohm's Law,

$$\vec{j} = \sigma \vec{E} \quad \text{bzw.} \quad \vec{j}_c = \sigma \vec{E}_c \tag{14}$$

with the electric conductivity, $\sigma = \text{const.}$

Now we shall analyze, under which conditions the displacement current can be neglected. To that end wie introduce a "typical length scale", ℓ, over which the fields vary significantly. Of course, ℓ is of the same order of magnitude as the geometric scales of the circuit.

Now we consider Maxwell's equations for the fields within the various building blocks of the circuit. Since outside of the conductors $\vec{j} = 0$, (11) becomes, using (4) with $\epsilon = \epsilon_0$, $\mu = \mu_0$

$$|\operatorname{curl} \vec{B}_c| = \mu_0 |\operatorname{curl} \vec{H}_c| = \epsilon_0 \mu_0 \omega |\vec{E}_c| = \frac{\omega}{c^2} |\vec{E}_c| \simeq \frac{1}{\ell} |\vec{B}_c|. \tag{15}$$

Thus we can neglect the displacement current, $\vec{j}_c = -\mathrm{i}\omega \vec{E}_c / c^2$, if

$$\frac{\omega}{c^2} |\vec{E}_c| \ll \frac{1}{\ell} |\vec{B}_c|. \tag{16}$$

By Faraday's law of induction (9) we have

$$\frac{1}{\ell} |\vec{E}_c| \simeq |\operatorname{curl} \vec{E}_c| = \omega |\vec{B}_c|. \tag{17}$$

Plugging this into (17), we get

$$\frac{\omega}{c^2} |\vec{E}_c| \simeq \frac{\omega^2 \ell}{c^2} |\vec{B}_c| \ll \frac{1}{\ell} |\vec{B}_c| \Rightarrow \frac{\omega \ell}{c} \ll 1, \tag{18}$$

i.e., the typical time scale for changes of the fields, $\tau \simeq 1/\omega$ should fullfil the condition

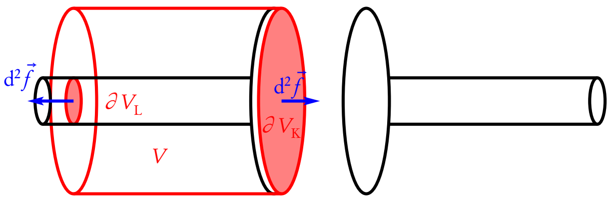
$$c\tau \gg \ell. \tag{19}$$

This means that the time a light signal needs to travel across the circuit should be very much smaller than the typical time scale τ for variations of the fields, i.e., these changes should be slow. An equivalent point of view is that retardation effects due to the finite speed of light can be neglected. Indeed, the introduction of the displacement current by Maxwell into the Ampére law lead to the prediction of **electromagnetic waves**, which finally have been discovered by H. Hertz in 1894.

Now we'll consider the interior of conductors (wires and resistances). Here we just have to estimate the order of magnitude of the displacement current in comparison to that of the charge-current densities:

$$|\vec{j}_{\text{D}c}| \ll |\vec{j}_c| \Rightarrow \epsilon_0 \omega |\vec{E}_c| \ll \sigma |\vec{E}_c| \Rightarrow \tau \gg \frac{\epsilon_0}{\sigma}. \tag{20}$$

Let's take copper as a common material for conductor, we have $\sigma = 59.8$ A/(Vm) at a temperature of 20°. This gives $\epsilon_0/\sigma = 1.48 \cdot 10^{-19}$s. For usual household current with $\tau = 2\pi/50$ Hz $= 0.13$ s thus (20) is very well fullfilled.



The only case, we *cannot* neglect the discplacement current is the space between the plates of capacitors. To see that, we consider the figure and apply Gauss's integral theorem to the volume, V, to the continuity equation,

$$\begin{aligned} &\operatorname{div}\vec{j} + \partial_t \rho = \operatorname{div}(\vec{j} + \epsilon_0 \partial_t \vec{E}) \\ \Rightarrow &\operatorname{div}(\vec{j}_c - \mathrm{i}\omega\epsilon_0 \vec{E}_c) = 0. \end{aligned} \tag{21}$$

Integration over the volume yields

$$-\mathrm{i}\omega\epsilon_0 \int_{\partial V_{\text{K}}} \mathrm{d}\vec{f} \cdot \vec{E} + \int_{\partial V_{\text{L}}} \mathrm{d}\vec{f} \cdot \vec{j} = 0. \tag{22}$$

As already established above, we can neglect the displacement current within the conductor. Thus, across the area ∂V_{L} only $\vec{j}$ is significant, i.e., $\vec{j}$ yields the complete current, I, through the conductor. Between the capacitor plates $\vec{j} = 0$, and thus we have to consider the displacement current in this region, because otherwise we'd get a contradiction with (22). The surface integral along the capacitor plate, ∂V_{K} yields the charge, Q, on the plate, and from (22) we get

$$I = \mathrm{i}\omega Q. \tag{23}$$

The same considerations give

$$|\vec{E}_{\text{K}c}| \simeq \frac{\sigma}{\epsilon_0\omega} \frac{F_{\text{L}}}{F_{\text{K}}} E_{\text{L}} \tag{24}$$

for the electric field $\vec{E}_{\text{K}}$ at the plates and across the conductor $\vec{E}_{\text{L}}$.

Although usually $F_{\text{L}} \ll F_{\text{K}}$, because of $\tau = 1/\omega \gg \epsilon_0/\sigma \simeq 10^{-18}$s ist, we still have

$$E_{\text{K}c} \gg E_{\text{L}c}. \tag{25}$$

Since the magneic field is continuous across the capacitor plate, $\vec{B}_{\text{K}c} \simeq \vec{B}_{\text{L}c}$ and thus due to Faraday's law of induction, $|\vec{E}_{\text{L}c}|/\ell = \omega |B_{\text{L}c}| \simeq \omega |B_{\text{K}}|$, we get

$$\omega |\vec{B}_{\text{K}c}| \simeq \omega |\vec{B}_{\text{L}}\, c| = \frac{1}{\ell} |\vec{E}_{\text{L}c}| \ll \frac{1}{\ell} |\vec{E}_{\text{K},c}|. \tag{26}$$

In the last step we have used (25). Thus we finally get

$$\omega|\vec{B}_{\mathrm{K}\,c}| \ll \frac{1}{\ell}|\vec{E}_{\mathrm{K},c}|, \tag{27}$$

i.e., between the capacitor plates we can neglect $\omega\vec{B}_c$ but not the displacement current.
The conclusion is that **between the capacitor plates** the **electro-quasistatic approximation**,

$$\operatorname{curl}\vec{E} = 0, \tag{28}$$

$$\operatorname{div}\vec{B} = 0, \tag{29}$$

$$\operatorname{curl}\vec{H} = \partial_t\vec{D}, \tag{30}$$

$$\operatorname{div}\vec{D} = 0, \tag{31}$$

holds. Thus we can treat the electric field between the capacitor plates as for an electrostatic field. Particularly, we have $Q = CU$, where U is the voltage at the capacitor. The continuity equation (22) thus yields

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \dot{Q} = C\dot{U} = \pm I, \tag{32}$$

where I is the total current within the wire connected to the capacitor plate. The sign is fixed by the usual sign conventions of circuit theory.
As we have seen, outside conductor plates, the **magneto-quasistatic approximation** holds, which are just Maxwell's equations with the displace current neglected,

$$\operatorname{curl}\vec{E} + \partial_t\vec{B} = 0, \tag{33}$$

$$\operatorname{div}\vec{B} = 0, \tag{34}$$

$$\operatorname{curl}\vec{H} = \vec{j}, \tag{35}$$

$$\operatorname{div}\vec{D}_c = \rho. \tag{36}$$

In addition within the wires and resistors (but of course not in inductances!) we can usually also neglect the magnetic field in (33), and thus integrating along a path through these elements of the circuit the DC laws hold at each instant of time, i.e., along wires and resistors Ohm's law is valid,

$$U = RI, \tag{37}$$

where U is the momentary voltage drop along the resister and I the total current running momentarily through it.

References

[Reb07] E. Rebhan, Theoretische Physik: Elektrodynamik, Springer Spektrum, Berlin, Heidelberg (2007).