Using (3.27) we obtain the Klein-Gordon equation for the real scalar field:

$$\Box\phi + m^2\phi = 0 \quad \text{with} \quad \Box = \partial_\mu\partial\mu = \partial_t^2 - \Delta. \tag{3.29}$$

Now we can interpret this equation with help of the meaning of the gradient operator known from nonrelativistic quantum mechanics to be the momentum operator: $\mathbf{p}_\mu = \mathrm{i}\partial_\mu$. So the Klein-Gordon equation gives the relativistic relation between energy and momentum of a free particle, namely $E^2 = \vec{p}^2 + m^2$.

To find a consistent definition of physical quantities we now prove Noether's theorem.

3.3 Noether's Theorem (Classical Part)

As shown above, the classical field theory is defined by an action functional $S[\phi]$. The physical fields are given by the condition of stationarity of S

$$\frac{\delta S}{\delta \phi} = 0, \tag{3.30}$$

which is the equation of motion for the fields. The action functional is given as the four-dimensional integral of the Lagrange density, which is a function of the fields ϕ and their gradients $\partial_\mu\phi$:

$$S[\phi] = \int \mathrm{d}^4x \mathscr{L}(\phi, \partial_\mu\phi). \tag{3.31}$$

The only constraint on the field is that it must vanish at infinity of four-dimensional space such that S exists.

Calculating the functional derivative with help of the techniques shown in appendix A shows that the stationarity condition (3.30) gives the Euler-Lagrange equations for the fields:

$$\frac{\delta S[\phi]}{\delta\phi} = \frac{\partial\mathscr{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} = 0. \tag{3.32}$$

Now we look at a rather general class of symmetry transformations which are described by the operation of a Lie group on the fields and space-time variables. The only assumption we want to make is that the action of the group on the space-time variables is independent on the fields.

Then the operation of an infinitesimal transformation can be described by

$$x'^\mu = x^\mu + \delta x^\mu, \quad \phi'(x') = \phi(x) + \delta\phi(x). \tag{3.33}$$

The field variation $\delta\phi(x)$ contains the operation of global and local internal symmetries of the fields as well as the action of the Lorentz group on the fields. Now we have to calculate the change of the action under the change of such an operation:

$$\delta S[\phi] = \int \mathrm{d}^4x' \mathscr{L}(\phi', \partial'_\mu\phi') - \int \mathrm{d}^4x \mathscr{L}(\phi, \partial_\mu\phi). \tag{3.34}$$

In the first integral we have to change the variables of integration to the original space-time variables. Up to first order in δx the Jacobian of the transformation is given by

$$\det\left(\frac{\partial x'^\mu}{\partial x_\nu}\right) = \det\left(\delta^\mu_\nu + \partial_\nu\delta x^\mu\right) = 1 + \partial_\mu\delta x^\mu. \tag{3.35}$$

This can be seen easily using the definition of a determinant as the sum over permutations of matrix elements. In first order in δx only the product of the diagonal elements is important. All other products appearing in the definition of the determinant contain at least two factors δx.

Now we have to take into account that variation and differentiation of the fields do not commute since we are varying the space-time variables as well as the fields:

$$\delta(\partial_\mu \phi) = \partial'_\mu \phi'(x') - \partial_\mu \phi = \partial_\nu(\phi + \delta\phi) \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)^{-1} - \partial_\mu \phi = \partial_\mu(\delta\phi) - (\partial_\mu \delta x^\nu)\partial_\nu \phi. \tag{3.36}$$

Taking (3.35) and (3.36) together with (3.34) we obtain after integrations by parts

$$\delta S[\phi] = \int \mathrm{d}^4 x \left[\frac{\delta S[\phi]}{\delta \phi(x)} \delta\phi + \partial_\mu \left((\partial_\nu \phi) \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} - \delta^\mu_\nu \mathscr{L}\right) \delta x^\nu\right] = 0. \tag{3.37}$$

The vanishing of the variation of the action functional for all fields (not only for solutions of the equations of motion!) is the definition for symmetry transformations. Now the identical vanishing of the integral for all fields can only be true if the integrand is a four-divergence.

A little calculation concerning the derivative in the second term gives

$$-\int \mathrm{d}^4 x \frac{\delta S[\phi]}{\delta \phi(x)} (\partial_\nu \phi) \delta x^\nu. \tag{3.38}$$

Now the infinitesimal operation of the group can be written in the form

$$\delta\phi(x) = \tau_a(x, \phi)\delta\eta^a, \quad \delta x^\mu = -T^\mu_a(x)\delta\eta^a, \tag{3.39}$$

where τ and T are bases of the Lie algebra of the group in the representation on the fields and space-time variables respectively. The $\delta\eta$ are real parameters independent on x and ϕ. All together we draw the conclusion that the integrand of (3.37) has to be a four-divergence:

$$\left[\frac{\delta S[\phi]}{\delta \phi(x)} \tau_a + \frac{\delta S[\phi]}{\delta \phi(x)} (\partial_\nu \phi) T^\nu_a\right] \delta\eta^a = -\partial_\mu j^\mu_a \delta\eta^a. \tag{3.40}$$

For the solutions of the field equations, i.e., such fields which fulfil the stationarity condition (3.30) we have

$$\partial_\mu j^\mu_a = 0 \tag{3.41}$$

since the generators of the Lie group T^a and τ^a are linearly independent.

This is *Emmy Noether's Theorem*:

For each generator of a symmetry group of the action functional there exists a current j^μ_a with vanishing four-divergence. These currents are called the *Noether currents* of the symmetry.

Now we have to find the explicit expressions for the currents. That means we have to express the vanishing of the four-divergence and the constraint on the group to be a symmetry of the action with help of the Lagrange density rather than with help of the action functional. Using (3.37 we find

$$-\delta\eta^a \partial_\mu j^\mu_a = \partial_\mu \left[\left(\partial_\nu \phi \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} - \delta^\mu_\nu \mathscr{L}\right) \delta x^\nu - \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \delta\phi\right] + \delta\mathscr{L} + \mathscr{L} \partial_\mu \delta x^\mu. \tag{3.42}$$

So we conclude that the group operation is a symmetry of the action, if there exists a field Ω such that

$$\exists \Omega^\mu_a(\phi, x): \quad \delta\mathscr{L} + \mathscr{L}\partial_\mu \delta x^\mu = -\partial_\mu \Omega^\mu_a \delta\eta^a, \tag{3.43}$$

and then the Noether currents are given by

$$\delta\eta^a j_a^\mu = -\left(\partial_\nu \phi \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} - \delta_\nu^\mu \mathscr{L}\right)\delta x^\nu + \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\delta\phi + \Omega_a^\mu \delta\eta^a. \tag{3.44}$$

Now we show that Noether's theorem is the local formulation of a conservation law. This can be done by integrating the vanishing four-divergence over an arbitrary four-dimensional volume V which has a boundary ∂V built by three dimensional space-like hypersurface of the four-dimensional space-time. Hereby a hypersurface is called space-like if all its normal vectors are time-like. Integrating (3.41) over this volume and applying the four-dimensional version of Gauss's integral theorem we obtain

$$\int_{\partial V} j_a^\mu \mathrm{d}S_\mu = 0. \tag{3.45}$$

Now a set of space-like hypersurfaces parameterised by $f(x) = \tau = \text{const}$ (with f an arbitrary scalar field with time-like gradients) defines an invariant measure of time τ. Now we assume that the four-volume V is bounded by the two hypersurfaces $S_1 : f(x) = \tau_1$ and $S_2 : f(x) = \tau_2$:

$$Q_a(\tau_1) = \int_{S_1} j_a^\mu dS_\mu = \int_{S_2} j_a^\mu dS_\mu = Q_a(\tau_2). \tag{3.46}$$

This means that the integral over the space-like hypersurface is a quantity constant in time and that the Qs are independent of the special choice of the space-like hypersurface. For convenience we may use the hypersurface $x^0 = t$ in a given reference frame:

$$Q_a = \int \mathrm{d}^3\vec{x}\, j_a^0(x). \tag{3.47}$$

The Q_a are called the *Noether charges* of the symmetry group.

Now we look at space-time translations. The four Noether charges of this group are the total energy and momentum of the fields. An infinitesimal translation in space and time is defined by

$$\delta\phi(x) = 0, \quad \delta x = -\delta a = \text{const} \Rightarrow \tau_a(x,\phi) = 0, \quad T_a^\mu(x) = \delta_a^\mu, \quad \delta\eta^a = -\delta a^a. \tag{3.48}$$

It is easy to see that the symmetry condition is fulfilled with setting $\Omega^\mu \cong 0$. Then, with help of (3.40), we obtain the corresponding Noether currents ${\Theta^\mu}_a$:

$${\Theta^\mu}_a = \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)}\right)\partial_a \phi - \delta_a^\mu \mathscr{L}. \tag{3.49}$$

This is the so called *canonical energy momentum tensor*, which has no direct physical meaning because it is not unique as we shall show in a moment. On the other hand, the Noether charges are unique physical quantities, namely total energy and momentum of the field system:

$$P_\nu(t) = \int \mathrm{d}^3\vec{x}{\Theta^0}_\nu. \tag{3.50}$$

The Noether currents can be changed with an arbitrary four-divergence without changing the Noether charges. With the j_a^μ defined above there are equivalent Noether currents given by

$${j'}^\mu_a = j_a^\mu + \partial_\nu k_a^{\nu\mu} \tag{3.51}$$

Since the divergence of this currents should vanish, we just have to take the $k_a^{\mu\nu}$ antisymmetric with respect of μ and ν, which is a rather weak assumption.

As we shall see, in the case of electrodynamics the canonical energy-momentum tensor cannot be interpreted as density of energy and momentum in all cases of physical interest. For instance in electrodynamics this tensor is not gauge invariant, but we shall see that we can define a physical energy-momentum tensor (the so called Belinfante tensor) which is gauge invariant and gives the well-known expressions for energy and momentum density in form of the familiar Poynting vector.

The spatial components of the physical energy-momentum tensor have the physical meaning of a tension. This can be seen by derivation of the total momentum (3.49) with respect to time and using its conservation. In the case of electrodynamics the space components of the physical energy momentum tensor is Maxwell's tensor of tension.

Now we apply Noether's theorem to the case of Lorentz transformations. An infinitesimal Lorentz transformation acts on the fields and space-time coordinates as follows:

$$\delta\phi(x) = \frac{1}{2}\delta\omega_{\mu\nu}\hat{\sigma}^{\mu\nu}\phi(x), \quad \delta x_\mu = \delta\omega_{\mu\nu}x^\nu, \tag{3.52}$$

where $\hat{\sigma}^{\mu\nu} = -\hat{\sigma}^{\nu\mu}$ are the six generators of the representation of the $\mathrm{SL}(2,\mathbb{C})$ which is the covering group of the $\mathrm{SO}(1,3)$. Because we have $\partial_\mu \delta x^\mu = 0$ the Lorentz invariance (which means that the Lorentz transformations are a symmetry group of the action) is the constraint on the Lagrange density to be a scalar field (seen as a function of x). The six Noether currents are then given by

$$J^{\rho,\mu\nu} = x^\mu\Theta^{\rho\nu} - x^\nu\Theta^{\rho\mu} - \frac{\partial\mathscr{L}}{\partial(\partial_\rho\phi)}\hat{\sigma}^{\mu\nu}\phi. \tag{3.53}$$

Here it is important to anti-symmetrise the currents $J^{\rho,\mu\nu}$ with respect to μ and ν since $\delta\omega^{\mu\nu}$ is antisymmetric. The μ and ν indices label the six Noether currents. Thus the six Noether charges are given by

$$J^{\mu\nu} = \int_{\partial V} \mathrm{d}S_\rho J^{\rho,\mu\nu}. \tag{3.54}$$

Herein V has the same meaning as in (3.45). The three space components coming from the invariance under rotations build therefore the total angular momentum of the system. By looking at (3.53) and the meaning of the energy-momentum tensor one sees that the angular-momentum tensor contains an orbital and a spin part. However, in relativistic physics there is no specific meaning to distinguish orbital and spin angular momentum. Only the total angular momentum has a definite physical meaning.

The conservation of the three mixed temporal-spatial components of $J^{\mu\nu}$ originates from the invariance under boost transformations. So this is the relativistic analogue of the centre of mass motion in the nonrelativistic case.

We close this section with the construction of a symmetric energy-momentum tensor. This is important for general relativity since there the energy-momentum tensor is necessarily symmetric. We shall see in the next chapter that in the case of electrodynamics the tensor can be chosen to be gauge-invariant, which is important to show that energy and momentum densities are sensible physical quantities in this case.

We start with (3.53) and the fact that it is conserved,

$$0 = \partial_\rho J^{\rho,\mu\nu} = \Theta^{\mu\nu} - \Theta^{\nu\mu} - \partial_\rho\frac{\partial\mathscr{L}}{\partial(\partial_\rho\phi)}\hat{\sigma}^{\mu\nu}\phi, \tag{3.55}$$

which shows that the canonical energy-momentum tensor is in general not symmetric in the indices μ and ν.

Now we make the ansatz

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho \omega^{\rho\mu\nu}, \tag{3.56}$$

where $\omega^{\rho\mu\nu}$ is an arbitrary tensor field which is antisymmetric in the indices ρ and μ.

We try to choose $\omega^{\rho\mu\nu}$ such that $T^{\mu\nu}$ is a symmetric tensor. Since it differs from the canonical energy-momentum tensor only by a total divergence it yields the same total energy and momentum for the field configuration. The antisymmetry of $\omega^{\rho\mu\nu}$ in ρ and μ makes the divergence $\partial_\mu T^{\mu\nu}$ vanish if $\partial_\mu \Theta^{\mu\nu} = 0$. This means $T^{\mu\nu}$ is as well a representant of the energy-momentum density as the canonical energy-momentum tensor $\Theta^{\mu\nu}$.

Inserting this ansatz into (3.55) shows that it is consistent with setting

$$\partial_\rho \left(\omega^{\rho\mu\nu} - \omega^{\rho\nu\mu}\right) = \partial_\rho \left[\frac{\partial \mathscr{L}}{\partial(\partial_\rho \phi)} \hat{\sigma}^{\mu\nu} \phi\right]. \tag{3.57}$$

The general solution of this equation is given by

$$\omega^{\rho\mu\nu} - \omega^{\rho\nu\mu} = \frac{\partial \mathscr{L}}{\partial(\partial_\rho \phi)} \hat{\sigma}^{\mu\nu} \phi + \partial_\sigma \eta^{\sigma\rho\mu\nu} := \eta^{\rho\mu\nu}, \tag{3.58}$$

where $\eta^{\sigma\rho\mu\nu}$ is an arbitrary tensor field which is antisymmetric in σ and ρ as well as in μ and ν. It is clear that then $\eta^{\rho\mu\nu}$ is antisymmetric in μ and ν.

Now using

$$\omega^{\rho\mu\nu} - \omega^{\rho\nu\mu} = \eta^{\rho\mu\nu}, \quad \omega^{\rho\mu\nu} + \omega^{\mu\rho\nu} = 0 \tag{3.59}$$

we find that with given $\eta^{\rho\mu\nu}$ (3.58) is solved uniquely by

$$\omega^{\rho\mu\nu} = \frac{1}{2}\left[\eta^{\rho\mu\nu} + \eta^{\mu\nu\rho} - \eta^{\nu\rho\mu}\right]. \tag{3.60}$$

It is easy to show by an algebraic calculation that indeed ω fulfills the conditions that we derived for it above. So we find the theorem, proven first by Belinfante in 1939 [Bel39], that we can always find a symmetric energy-momentum tensor.

We shall see in the next chapter that by a clever choice of $\eta^{\sigma\rho\mu\nu}$ which is the only freedom we have to make the energy-momentum tensor symmetric, makes the energy-momentum tensor of the electromagnetic field also gauge invariant.

3.4 Canonical Quantisation

Now we like to solve our problem with the particle interpretation and causality raised by the negative energy states. For this purpose let us consider a free complex scalar field with the Lagrangian

$$\mathscr{L} = (\partial_\mu \phi)^* (\partial^\mu \phi) - m^2 \phi^* \phi. \tag{3.61}$$

Although there seems to be no solution in terms of a Schrödinger-like theory, i.e., to interpret the ϕ-field as a one-particle wave function, we try to build a many-particle theory by quantising the fields.

from: M.E. Peskin, D.V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley 1995

Goldstone's Theorem

The appearance of massless particles when a continuous symmetry is spontaneously broken is a general result, known as *Goldstone's theorem.* To state the theorem precisely, we must count the number of linearly independent continuous symmetry transformations. In the linear sigma model, there are no continuous symmetries for $N = 1$, while for $N = 2$ there is a single direction of rotation. A rotation in N dimensions can be in any one of $N(N-1)/2$ planes, so the $O(N)$-symmetric theory has $N(N-1)/2$ continuous symmetries. After spontaneous symmetry breaking there are $(N-1)(N-2)/2$ remaining symmetries, corresponding to rotations of the $(N-1)$ π fields. The number of *broken* symmetries is the difference, $N-1$.

Goldstone's theorem states that for every spontaneously broken continuous symmetry, the theory must contain a massless particle.† We have just seen that this theorem holds in the linear sigma model, at least at the classical level. The massless fields that arise through spontaneous symmetry breaking are called *Goldstone bosons.* Many light bosons seen in physics, such as the pions, may be interpreted (at least approximately) as Goldstone bosons. We conclude this section with a general proof of Goldstone's theorem for classical scalar field theories. The rest of this chapter is devoted to the quantum-mechanical analysis of theories with hidden symmetry. By the end of the chapter we will see that Goldstone bosons cannot acquire mass from any order of quantum corrections.

Consider, then, a theory involving several fields $\phi^a(x)$, with a Lagrangian of the form

$$\mathcal{L} = (\text{terms with derivatives}) - V(\phi). \tag{11.10}$$

Let ϕ_0^a be a constant field that minimizes V, so that

$$\frac{\partial}{\partial\phi^a} V\bigg|_{\phi^a(x)=\phi_0^a} = 0.$$

Expanding V about this minimum, we find

$$V(\phi) = V(\phi_0) + \frac{1}{2}(\phi-\phi_0)^a(\phi-\phi_0)^b\left(\frac{\partial^2}{\partial\phi^a\partial\phi^b}V\right)_{\phi_0} + \cdots .$$

The coefficient of the quadratic term,

$$\left(\frac{\partial^2}{\partial\phi^a\partial\phi^b}V\right)_{\phi_0} = m_{ab}^2, \tag{11.11}$$

†J. Goldstone, *Nuovo Cim.* **19**, 154 (1961). An instructive four-page paper by J. Goldstone, A. Salam, and S. Weinberg, *Phys. Rev.* **127**, 965 (1962), gives three different proofs of the theorem.

is a symmetric matrix whose eigenvalues give the masses of the fields. These eigenvalues cannot be negative, since ϕ_0 is a minimum. To prove Goldstone's theorem, we must show that every continuous symmetry of the Lagrangian (11.10) that is not a symmetry of ϕ_0 gives rise to a zero eigenvalue of this mass matrix.

A general continuous symmetry transformation has the form

$$\phi^a \longrightarrow \phi^a + \alpha\Delta^a(\phi), \tag{11.12}$$

where α is an infinitesimal parameter and Δ^a is some function of all the ϕ's. Specialize to constant fields; then the derivative terms in $\mathcal{L}$ vanish and the potential alone must be invariant under (11.12). This condition can be written

$$V(\phi^a) = V\big(\phi^a + \alpha\Delta^a(\phi)\big) \qquad \text{or} \qquad \Delta^a(\phi)\frac{\partial}{\partial\phi^a}V(\phi) = 0.$$

Now differentiate with respect to ϕ^b, and set $\phi = \phi_0$:

$$0 = \left(\frac{\partial\Delta^a}{\partial\phi^b}\right)_{\phi_0}\left(\frac{\partial V}{\partial\phi^a}\right)_{\phi_0} + \Delta^a(\phi_0)\left(\frac{\partial^2}{\partial\phi^a\partial\phi^b}V\right)_{\phi_0}. \tag{11.13}$$

The first term vanishes since ϕ_0 is a minimum of V, so the second term must also vanish. If the transformation leaves ϕ_0 unchanged (i.e., if the symmetry is respected by the ground state), then $\Delta^a(\phi_0) = 0$ and this relation is trivial. A spontaneously broken symmetry is precisely one for which $\Delta^a(\phi_0) \neq 0$; in this case $\Delta^a(\phi_0)$ is our desired vector with eigenvalue zero, so Goldstone's theorem is proved.

11.2 Renormalization and Symmetry: An Explicit Example

Now let us investigate the quantum mechanics of a theory with spontaneously broken symmetry. Again we will use as our example the linear sigma model. The Lagrangian of this theory, written in terms of shifted fields, is given in Eq. (11.9). From this expression, we can read off the Feynman rules; these are shown in Fig. 11.3.

Using these Feynman rules, we can compute tree-level amplitudes without difficulty. Diagrams with loops, however, will often diverge. For the amplitude with N_e external legs, the superficial degree of divergence is

$$D = 4 - N_e,$$

just as in the discussion of ϕ^4 theory in Section 10.2. (Diagrams containing a three-point vertex will be less divergent than this expression indicates, because this vertex has a coefficient with dimensions of mass.) However, the symmetry constraints on the amplitudes are much weaker than in that earlier analysis. The linear sigma model has eight different superficially divergent amplitudes (see Fig. 11.4); several of these have $D > 0$ and therefore can contain