S. Leupold, H. van Hees **Winter Semester 2008/9**

Exercises for Higher Quantum Mechanics

Deadline: Oct/29/2008

Sheet 1

Question 1 (Spin 1/2)

The spin matrices for spin, $s = 1/2$, in the standard representation, where $\mathbf{s}_3$ is diagonal, read

$$\mathbf{s}_1 = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{s}_2 = \frac{\hbar}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \mathbf{s}_3 = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

(a) Show that these spin matrices fulfill the commutator relations for angular momenta,

$$[\mathbf{s}_a, \mathbf{s}_b] = \mathrm{i}\hbar\epsilon_{abc}\mathbf{s}_c. \tag{2}$$

Herein ϵ_{abc} denotes the Levi-Civita symbol which is completely anti-symmetric under interchanges of its indices, a, b, and c with $\epsilon_{123} = 1$. In (2), summation over the repeated index, c, is understood (Einstein's summation convention). Check that in this basis also the squared modulus of the spin, $\vec{\mathbf{s}}^2$ is diagonal simultaneously. What are the corresponding eigenvalues of this operator?

(b) Calculate the eigenvalues and eigenvectors of the operator, $\mathbf{s}_1$.

(c) Suppose an electron is prepared with a Stern-Gerlach apparatus such that the s_3 component of its spin has the value $\sigma_3 = +\hbar/2$. What is the probability to find the value $\sigma_1 = -\hbar/2$ for the s_1-spin component?

(d) Calculate the unitary operator

$$\mathbf{U}_3(\varphi) = \exp\left(-\frac{\mathrm{i}}{\hbar}\varphi\mathbf{s}_3\right). \tag{3}$$

(e) Let $\vec{n} \in \mathbb{R}^3$ denote an arbitrary unit vector ($|\vec{n}| = 1$) and

$$\vec{\boldsymbol{\sigma}} = \frac{2}{\hbar}\vec{\mathbf{s}} \tag{4}$$

Pauli's spin matrices.

Show that

$$\mathbf{U}_{\vec{n}}(\varphi) := \exp\left(-\mathrm{i}\frac{\varphi}{2}\vec{n}\vec{\boldsymbol{\sigma}}\right) = \mathbf{1}\cos\left(\frac{\varphi}{2}\right) - \mathrm{i}\vec{n}\vec{\boldsymbol{\sigma}}\sin\left(\frac{\varphi}{2}\right) \tag{5}$$

and that this is a unitary operator.

Hints for part (e): To calculate the operator exponential one has to use the series expansion in powers of the operator (see lecture notes). First show the relation,

$$(\vec{n}\vec{\boldsymbol{\sigma}})^2 = \mathbf{1}. \tag{6}$$

Then one needs the power expansion for cos and sin:

$$\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}, \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}. \end{aligned} \tag{7}$$

Remark: $\mathbf{U}_{\vec{n}}(\varphi)$ corresponds to rotations in $\mathbb{R}^3$ in the following sense. If one defines for an arbitrary $\vec{x} \in \mathbb{R}^3$ the operator Operator

$$\tilde{\mathbf{x}} = \vec{x}\vec{\boldsymbol{\sigma}}, \tag{8}$$

one can show with help of Eq. (5) that

$$\mathbf{U}_{\vec{n}}(\varphi)\tilde{\mathbf{x}}\mathbf{U}^{\dagger}_{\vec{n}}(\varphi) = \tilde{\mathbf{x}'} = \vec{x}\,'\vec{\boldsymbol{\sigma}} \tag{9}$$

with

$$\vec{x}\,' = (\vec{n}\vec{x})\vec{n} + [\vec{x} - (\vec{n}\vec{x})\vec{n}]\cos\varphi + (\vec{n} \times \vec{x})\sin\varphi \tag{10}$$

holds, i.e., $\vec{x}\,'$ results from a rotation of the vector, $\vec{x}$, by an rotation angle φ around the axis, $\vec{n}$ (in the sense of the right-hand rule).

Suggested reading:

J. J. Sakurai, Modern Quantum Mechanics, Addison Wesley
E. Fick, Einführung in die Grundlagen der Quantentheorie, Aula-Verlag
L. D. Landau, E. M. Lifschitz, Quantum Mechanics, Pergamon Press