S. Leupold, H. van Hees **Winter Semester 2008/9**

Exercises for Advanced Quantum Mechanics

Deadline: Nov/4/2008

Sheet 2

Question 1 (Fourier transformation and Dirac's δ distribution)
We consider a particle, which moves along one spatial dimension. As shown in the lecture, the spectrum of position and momentum operators, $\mathbf{x}$ and $\mathbf{p}$, is continuous, i.e., all real numbers are "generalized eigenvalues". The corresponding generalized eigenfunctions have been normalized such that the completeness relations read

$$\int_{-\infty}^{\infty} \mathrm{d}x \, |x\rangle \langle x| = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi\hbar} |p\rangle \langle p| = \mathbb{1}. \tag{1}$$

Further the momentum eigenfunction in the position representation has come out to be

$$\langle x \,|\, p \rangle = \exp\left(\frac{\mathrm{i}px}{\hbar}\right). \tag{2}$$

The completeness relation for the momentum eigenstates (1) in the position representation thus reads

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi\hbar} \langle x \,|\, p \rangle \langle p \,|\, x' \rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi\hbar} \exp\left[\frac{\mathrm{i}p(x-x')}{\hbar}\right] = \langle x \,|\, x' \rangle = \delta(x-x'), \tag{3}$$

where δ is the Dirac δ distribution.

(a) Show that because of the completeness relation of the generalized position eigenstates the relation

$$\int_{-\infty}^{\infty} \mathrm{d}x \exp\left[\frac{\mathrm{i}x(p'-p)}{\hbar}\right] = 2\pi\hbar\delta(p'-p) \tag{4}$$

holds true. Motivate with help of (4) that

$$\delta(p-p') = \delta(p'-p). \tag{5}$$

(b) Let $|\psi\rangle$ be a Hilbert-space vector. Show that the corresponding wave functions in the position representation, $\psi(x) := \langle x \,|\, \psi \rangle$, and momentum representation, $\tilde{\psi}(p) = \langle p \,|\, \psi \rangle$, are related to each other by the Fourier transformations,

$$\psi(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi\hbar} \exp\left(\frac{\mathrm{i}px}{\hbar}\right) \tilde{\psi}(p), \quad \tilde{\psi}(p) = \int_{-\infty}^{\infty} \mathrm{d}x \exp\left(-\frac{\mathrm{i}px}{\hbar}\right) \psi(x). \tag{6}$$

(c) Show that the scalar products of Hilbert-space vectors are independent of the choice of representation, i.e., that for two Hilbert-space vectors, $|\psi_1\rangle$ and $|\psi\rangle_2$

$$\langle \psi_1 \,|\, \psi_2 \rangle = \int_{-\infty}^{\infty} \mathrm{d}x \psi_1^*(x)\psi_2(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi\hbar} \tilde{\psi}_1^*(p)\tilde{\psi}_2(p). \tag{7}$$

(d) The state of a particle is given by the position-wave function,

$$\psi(x) = N \exp\left(-\frac{x^2}{4\Delta x^2}\right). \tag{8}$$

Determine the normalization constant such that the normalization condition $\langle \psi \,|\, \psi \rangle = 1$ is fulfilled. Determine the momentum-wave function as well as the expectation values and standard deviations of position and momentum. Establish the independence of the expectation value of the momentum from the representation, used to calculate it, i.e, that indeed

$$\langle \psi \,|\mathbf{p}|\, \psi \rangle = \int_{-\infty}^{\infty} \mathrm{d}x \psi^*(x) \left(\frac{\hbar}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}x}\right) \psi(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi\hbar} \tilde{\psi}^*(p) p \tilde{\psi}(p). \tag{9}$$

Hint: You can use the Gauss integral,

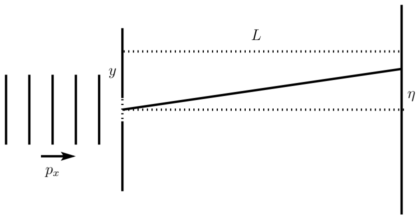
$$G(a) = \int_{-\infty}^{\infty} \mathrm{d}x \exp(-ax^2) = \sqrt{\frac{\pi}{a}}, \tag{10}$$

without proof. Note that, by differentiation with respect to a, you obtain the integrals,

$$\int_{-\infty}^{\infty} \mathrm{d}x \, x^{2n} \exp(-ax^2) = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}a^n} G(a). \tag{11}$$

Please, find Question 2 on the next page!

Question 2 (Movement of a particle through a slit, uncertainty relation)



A stream of particles (e.g., electrons) with momentum $\vec{p} = p_x \hat{e}_x$ hits a slit of width, d. Under the assumption that the slit is hit uniformly by the particles, the position-wave function of the particles at the moment they hit the slit is given by

$$\psi(x, y) = \frac{1}{\sqrt{2\pi d}} \Theta\left(\frac{d}{2} - |y|\right) \exp\left(\frac{\mathrm{i} x p_x}{\hbar}\right). \tag{12}$$

Here,

$$\Theta(\xi) = \begin{cases} 1 & \text{for } \xi \geq 1 \\ 0 & \text{for } \xi < 1 \end{cases} \tag{13}$$

denotes the Heaviside-unit-step function. The particles are registered with help of a photo plate at a distance L from the slot.

(a) Determine the probability distribution, $P(p_y)$, for the momentum component, p_y.

(b) Estimate the uncertainty, Δp_y, of the momentum component, p_y, by the width of the main maximum of the probability distribution, $P(p_y)$.

(c) Interpret the result, considering the Heisenberg-uncertainty relation for the preparation of position and momentum of the particle.

(d) The momentum, p_y, is measured by the position η at the photo plate. Derive the corresponding distribution $\tilde{P}(\eta)$! You may assume that the momentum component p_x is undisturbed when the particle moves through the slit.