Vorkurs Mathematik, Blatt 3

(1) $f(x) = 1 + |x|$

Stetigkeit bei $x=0$. Prüfe rechts und linksseitigen Grenzwert für

$x = 0$:

$$\lim_{x\to 0^+} f(x) = \lim_{x \to 0^+} 1+x = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(-x) = \lim_{x\to 0^+} (1+x) = 1$$

$\Rightarrow f$ stetig bei $x=0$

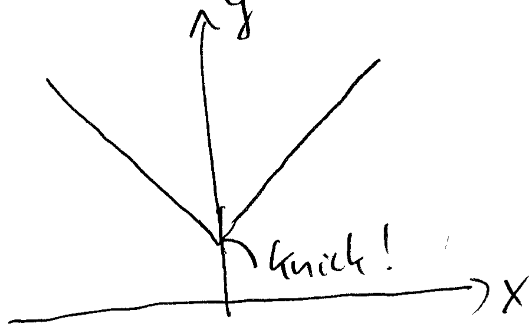
Ableitung bei $x=0$: rechtsseitig $(\Delta x > 0)$

$$\frac{f(\Delta x) - f(0)}{\Delta x} \overset{\Delta x > 0}{=} \frac{1+\Delta x - 1}{\Delta x} = 1 \xrightarrow[\Delta x \to 0^+]{} 1$$

linksseitig $(\Delta x < 0)$

$$\frac{f(\Delta x) - f(0)}{\Delta x} \overset{\Delta x < 0}{=} \frac{1 - \Delta x - 1}{\Delta x} = -1 \xrightarrow[\Delta x \to 0^-]{} -1$$

f ist nicht differenzierbar in $x=0$.



(2) Sekante: $x_1 = 1$; $x_2 = \frac{3}{2}$

$$y_1 = f(x_1) = 1^3 - 2 \cdot 1 = 1 - 2 = -1$$

$$y_2 = f(x_2) = \left(\frac{3}{2}\right)^3 - 2 \cdot \frac{3}{2} = \frac{27}{8} - 3 = \frac{3}{8}$$

Steigung: $m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{-1 - \frac{3}{8}}{1 - \frac{3}{2}} = \dfrac{-\frac{11}{8}}{-\frac{1}{2}} = \frac{11}{4}$

Steigung der Tangente bei x_1 ist

$$f'(x_1) = 3x_1^2 - 2 = 3 \cdot 1^2 - 2 = 1$$

(3) $g(x) = \frac{2}{3}x^3 - 2x^2 - 6x$

Nullstellen:

$$\frac{2}{3}x^3 - 2x^2 - 6x = 0$$

$$\Rightarrow x^3 - 3x^2 - 9x = 0$$

$$= x(x^2 - 3x - 9) = 0$$

Eine Nullstelle ist $x_1 = 0$. Für die beiden anderen verwende ich die "pq-Formel"

$x^2 - 3x - 9 = 0 \Rightarrow x_{2/3} = -\left(-\frac{3}{2}\right) \pm \sqrt{\frac{9}{4} + 9}$

$$= \frac{3}{2} \pm \sqrt{\frac{45}{4}} = \frac{3}{2} \pm \frac{3}{2}\sqrt{5}$$

Extrema

$$g'(x) = 2x^2 - 4x - 6$$

Nullstellen der Ableitung: $2x^2 - 4x - 6 = 0 \Rightarrow x^2 - 2x - 3 = 0$

$$x_{1/2} = -\left(-\frac{2}{2}\right) \pm \sqrt{1+3} = 1 \pm 2 = \begin{cases} 3 \\ -1 \end{cases}$$

$g''(x) = 4x-4$

$g''(x_{e1}) = 4\cdot 3-4 = 8 \Rightarrow$ lokales Minimum

$g''(x_{e2}) = 4\cdot(-1)-4 = -8 \Rightarrow$ lokales Maximum

Werte: $g(3) = \frac{2}{3}\cdot 3^3 - 2\cdot 3^2 - 6\cdot 3 = \underline{\underline{-18}}$

$g(-1) = \frac{2}{3}(-1)^3 - 2(-1)^2 - 6\cdot(-1) = -\frac{2}{3} - 2 + 6$

$= 4 - \frac{2}{3} = \frac{12-2}{3} = \underline{\underline{\frac{10}{3}}}$

$h(\varphi) = 2\cos(\varphi+2)$

Nullstellen: $\cos(\varphi+2) = 0 \Rightarrow \varphi+2 = \frac{\pi}{2} + k\pi \;;\; k\in\mathbb{Z}$

$\varphi = (k+\frac{1}{2})\pi - 2 \Rightarrow \varphi_1 = \frac{3}{2}\pi - 2 \;;\; \varphi_2 = \frac{5}{2}\pi - 2$

Extrema

$h'(\varphi) = -2\sin(\varphi+2) \stackrel{!}{=} 0 \Rightarrow \varphi+2 = k\pi \;;\; \pi\in\mathbb{Z}$

$\Rightarrow \varphi_{e1} = \pi-2 \;;\; \varphi_{e2} = 2\pi-2$

$h''(\varphi) = -2\cos(\varphi+2) \Rightarrow h''(\varphi_{e1}) = -2\cos\pi = \underline{\underline{2}} > 0$

$\Rightarrow$ lokales Minimum

$h''(\varphi_{e2}) = -2\cos(2\pi) = \underline{\underline{-2}} < 0 \Rightarrow$ lokales Maximum

Funktionswerte

$h(\varphi_{e1}) = 2\cos\pi = \underline{\underline{-2}} \;;\; h(\varphi_{e2}) = 2\cos(2\pi) = \underline{\underline{2}}$

4 a) $f(x) = \sqrt{a^2+x^2} + x^{-7/2}$ (4)

$$f'(x) = \frac{2x}{2\sqrt{a^2+x^2}} - \frac{7}{2}x^{-9/2} = \frac{x}{\sqrt{a^2+x^2}} - \frac{7}{2}x^{-9/2}$$

(b) $f(x) = (a+bx^m)^n = n(a+bx^m)^{n-1}\, mbx^{m-1}$

(c) $f(x) = \left(\frac{x^2}{1+x^2}\right)^n = \left(\frac{x^2+1-1}{x^2+1}\right)^n = \left(1-\frac{1}{x^2+1}\right)^n$

$$\Rightarrow f'(x) = n\left(1-\frac{1}{x^2+1}\right)^{n-1} \cdot (-1)\,\frac{2x(-1)}{(x^2+1)^2}$$

$$= n\left(\frac{x^2}{x^2+1}\right)^{n-1} \frac{2x}{(x^2+1)^2} = 2n\,\frac{x^{2n-1}}{(x^2+1)^{2n+1}}$$

(d) $f(x) = \ln\left[e^{g(x)}\right] = g(x)$

$$\Rightarrow f'(x) = g'(x)$$

(e) $f(x) = \left[1-\cos^2(bx)\right]^n \sin^n(bx)$

$$= \sin^{3n}(bx)$$

$$\Rightarrow f'(x) = 3n\sin^{3n-1}(bx) \cdot b\cos(bx)$$

$$= 3nb\cos(bx)\sin^{3n-1}(bx)$$

(5)

(4) $\frac{d}{dx}\cosh x = \frac{d}{dx}\frac{e^x+e^{-x}}{2} = \frac{1}{2}(e^x - e^{-x}) = \underline{\underline{\sinh x}}$

$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x-e^{-x}}{2} = \frac{1}{2}(e^x+e^{-x}) = \underline{\underline{\cosh x}}$

$\Rightarrow \frac{d^2}{dx^2}\cosh x = \underline{\underline{\cosh x}}$; $\frac{d^2}{dx^2}\sinh x = \underline{\underline{\sinh x}}$

$\frac{d}{dx}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \underline{\underline{\frac{1}{\cosh^2 x}}}$

$\frac{d^2}{dx^2}\tanh x = -2\cosh^{-3} x \cdot \sinh x = -\frac{2\sinh x}{\cosh^3 x}$