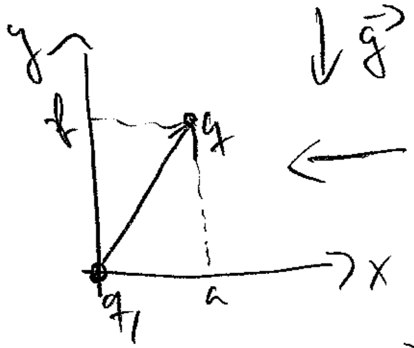
Exam I solutions (Problem 4 updated on Mar/08/07) ①

(1)



$\vec{E} = E\vec{i}_x$

$\vec{r} = a\vec{i}_x + b\vec{i}_y$

$$\vec{F}_{\text{on } q} = \frac{q q_1}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3} - mg\vec{i}_y + qE\vec{i}_x$$

$$F_x = \frac{q q_1}{4\pi\varepsilon_0}\frac{a}{(a^2+b^2)^{3/2}} + qE$$

$$F_y = \frac{q q_1}{4\pi\varepsilon_0}\frac{b}{(a^2+b^2)^{3/2}} - mg$$

Particle at rest $\Rightarrow F_x = F_y = 0$

$$F_y = 0 \Rightarrow q_1 = \frac{4\pi\varepsilon_0 mg(a^2+b^2)^{3/2}}{qb}$$

$$F_x = 0 \Rightarrow E = -\frac{q_1 a}{4\pi\varepsilon_0(a^2+b^2)^{3/2}} = -\frac{mga}{qb}$$

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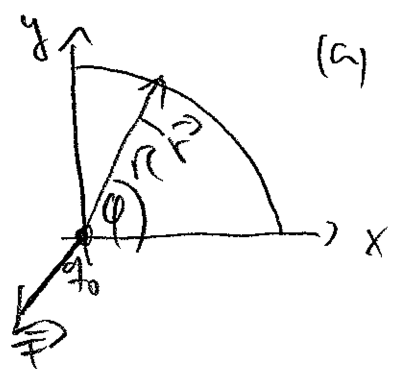
②



(a) $$dQ = \frac{Q}{\pi/2}d\varphi = \frac{2Q}{\pi}d\varphi$$

$$\vec{F} = \int_0^{\pi/2}\frac{2Q}{\pi}d\varphi\cdot\frac{q_0}{4\pi\varepsilon_0 R^2}(-\vec{i}_r)$$

$$= \frac{Qq_0}{2\pi^2\varepsilon_0 R^2}\int_0^{\pi/2}d\varphi(-\vec{i}_x\cos\varphi - \vec{i}_y\sin\varphi)$$

$$\vec{F} = -\frac{q_0 Q}{2\pi^2\varepsilon_0 R^2}(\vec{i}_x + \vec{i}_y)$$

(b) $$\vec{F} = -\int_0^{\pi/2} \underbrace{d\vartheta\, R\, \lambda(\vartheta)}_{dQ} \frac{q_0 \vec{i}_r}{4\pi\varepsilon_0 R^2}$$

$$= -\frac{q_0}{4\pi\varepsilon_0 R} \int_0^{\pi/2} d\vartheta \left(\cos\vartheta\, \vec{i}_x + \sin\vartheta\, \vec{i}_y\right) \lambda(\vartheta)$$

(3) (a) $$\vec{E} = \frac{C}{r^4} \vec{i}_r$$

$$\vec{E} = -\frac{\partial V}{\partial r} \vec{i}_r \quad \Rightarrow V = V(r)$$

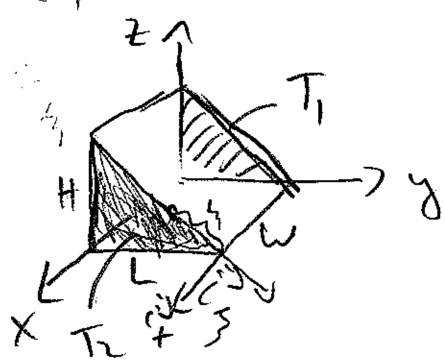
$$\frac{\partial V}{\partial r} = -\frac{C}{r^4} \Rightarrow V = +\frac{C}{3r^3}$$

(b) $$V(x,y) = \frac{C}{3(x^2+y^2)^{3/2}} = \frac{C}{3}(x^2+y^2)^{-3/2}$$

$$E_x = -\frac{\partial V}{\partial x} = \frac{Cx}{(x^2+y^2)^{5/2}}$$

$$E_y = -\frac{\partial V}{\partial y} = \frac{Cy}{(x^2+y^2)^{5/2}}$$

(4) (a)



T_1: $d\vec{S} = dy\,dz\,(-\vec{i}_x)$

$\vec{r} = y\vec{i}_y + z\vec{i}_z$

$y \in (0, L)$

For each y: $z \in (0, \frac{H}{L} y)$

I consider this as y in the parametrization!

$\vec{E}(\vec{r}) = \beta \vec{i}_y$ (because $x = 0$)

$\Rightarrow \Phi^{(1)}_{T_1} = \int_{T_1} d\vec{S} \cdot \vec{E} = 0$ because $d\vec{S} \cdot \vec{E} = 0$ everywhere

T_2: $d\vec{S} = dy\,dz\,\vec{i}_x$

$\vec{r} = W\vec{i}_x + y\vec{i}_y + z\vec{i}_z$

$y \in (0, L)$

for each y: $z \in (0, \frac{H}{L} y)$

$\vec{E}(\vec{r}) = \alpha W^2 \vec{i}_x + \beta \vec{i}_y$

$$\Phi^{(1)}_{T_2} = \int_{T_2} d\vec{S} \cdot \vec{E} = \int_0^L dy \int_0^{\frac{H}{L}y} dz\, \alpha W^2 = \int_0^L \alpha W^2 \frac{H}{L} y$$

$$= \alpha \frac{W^2 L H}{2}$$

(b) Sloped rectangular surface, S:

$$\vec r = x\,\vec i_x + \xi\,\vec i_\xi \;;\quad x \in (0, w),\quad \xi \in \left(0, \sqrt{L^2+H^2}\right)$$

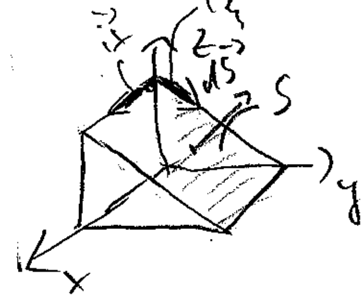
$$\vec i_\xi = \frac{L\,\vec i_y - H\,\vec i_z}{\sqrt{L^2+H^2}}$$

$$d\vec S = \frac{\partial \vec r}{\partial x} \times \frac{\partial \vec r}{\partial \xi}\, dx\, d\xi$$

chosen such that it points out of the volume (right-hand rule!)

$$= \vec i_x \times \vec i_\xi = \frac{1}{\sqrt{L^2+H^2}}\, \vec i_x \times \left(L\,\vec i_y - H\,\vec i_z\right) dx\, d\xi$$

$$= \frac{1}{\sqrt{L^2+H^2}} \left(L\,\vec i_z + H\,\vec i_y\right) dx\, d\xi$$



$$\vec E(\vec r) = \alpha\, x^2\, \vec i_x + \beta\, \vec i_y$$

$$d\vec S \cdot \vec E(\vec r) = \frac{\beta H}{\sqrt{L^2+H^2}}\, dx\, d\xi$$

$$\Rightarrow \Phi_S^{(1)} = \int_S d\vec S \cdot \vec E(\vec r) = \frac{\beta H}{\sqrt{L^2+H^2}} \int_0^w dx \int_0^{\sqrt{L^2+H^2}} d\xi \cdot 1$$

$$= \frac{\beta H}{\sqrt{L^2+H^2}}\, w \sqrt{L^2+H^2} = \beta H w$$

(c) That's surface T_2 already parametrized in (a)

But now

$$\vec E(\vec r) = \alpha\, z^2\, \vec i_x + \beta\, \vec i_y$$

$$\Phi_{T_2}^{(1)} = \int_0^L dy \int_0^{\frac{H}{L} y} dz\, \alpha z^2$$

$$\Phi_{T_2}^{(1)} = \frac{\alpha}{3}\left(\frac{H}{L}\right)^3 \int_0^L dy\, y^3$$

$$= \frac{\alpha}{12}\, \frac{H^3}{L^3}\, L^4$$

$$\Phi_{T_2}^{(1)} = \frac{\alpha}{12}\, H^3 L$$