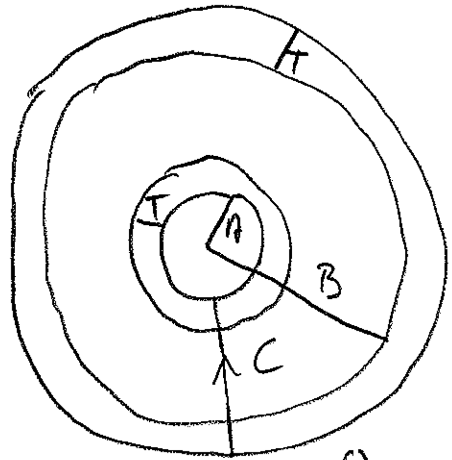
Exam II Solutions

(1)



(a) $$\sigma(\vec{r}) = \sigma(r) = \begin{cases} \dfrac{Q}{4\pi(A+T)^2} & \text{for } r = A+T \\ -\dfrac{Q}{4\pi B^2} & \text{for } r = B \\ \dfrac{Q}{4\pi (B+T)^2} & \text{for } r = B+T \end{cases}$$

This can be shown with help of Gauss's law and the symmetry of the electrostatic field

$$\vec{E}(\vec{r}) = -\operatorname{grad} V = -\frac{\partial V}{\partial r}\vec{i}_r \Rightarrow \vec{E}(\vec{r}) = E_r(r)\,\vec{i}_r$$

and the condition that in the static case $\vec{E} \equiv 0$ inside conductors.

(b) With Gauss's law we find

$$E_r(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \times \begin{cases} 0 & \text{for } 0 \le r < A+T \\ 1 & \text{for } A+T < r < B \\ 0 & \text{for } B < r < B+T \\ 1 & \text{for } B+T < r \end{cases}$$

$$\Delta V = -\int_C d\vec{r}\cdot\vec{E}(\vec{r}) = +\int_A^{B+T} dr\, E_r(r) = \int_{A+T}^{B} dr\,\frac{Q}{4\pi\varepsilon_0 r^2}$$

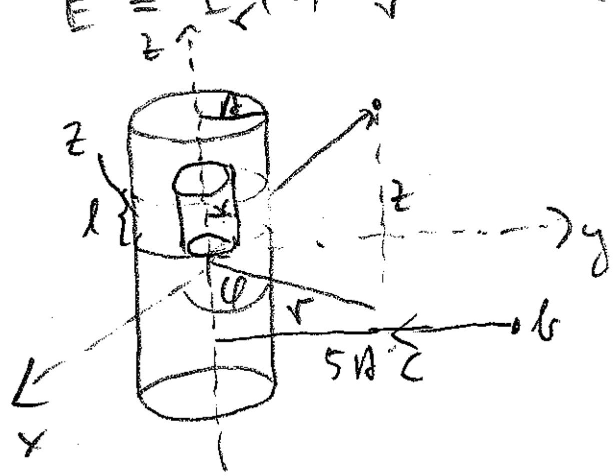
$$\Delta V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{A+T} - \frac{1}{B}\right)$$

(c) $$C = \frac{Q}{\Delta V} = \frac{4\pi\varepsilon_0}{\frac{1}{A+T} - \frac{1}{B}} = \frac{4\pi\varepsilon_0 (A+T) B}{B-(A+T)}$$

(2) (a) From the cylinder symmetry we know

$$\vec{E} = E_r(r)\,\vec{i}_r$$

(here, r is the radius of cylinder coordinates, i.e., $r = \sqrt{x^2+y^2}$!)



Take a cylinder Z as Gaussian surface. Then, if

<u>$r < A$:</u>

$$\oint_{\partial V} d\vec{S}\,\vec{E}(\vec{r}) = 2\pi r \int_0^l dl\; E_r(r) = 2\pi l r E_r(r)$$

$$= \frac{\rho\,\pi r^2 l}{\varepsilon_0}$$

$$\Rightarrow E_r(r) = \frac{\rho}{2\varepsilon_0} r$$

With $\rho = \dfrac{Q}{\pi A^2 L}$

$$E_r(r) = \frac{Q}{2\pi\varepsilon_0 A^2 L} r \quad (r < A)$$

For $r > A$ the LHS of Gauss's law stays the same, but the RHS is

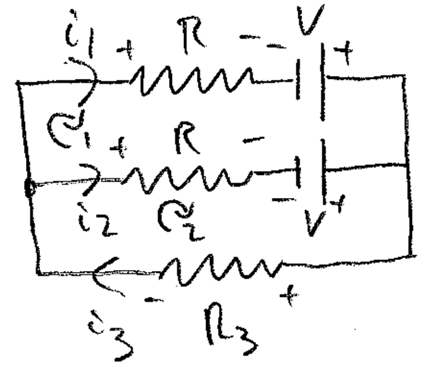
$$2\pi l r E_r(r) = \frac{\rho \pi A^2 l}{\varepsilon_0} = \frac{Q l}{\varepsilon_0 L}$$

$$\Rightarrow \quad E_r(r) = \frac{Q}{2\pi \varepsilon_0 L r} \qquad (r > A)$$

$$\Delta V = + \int_0^{5A} dr\, E_r(r) = + \int_0^{A} \frac{Q}{2\pi \varepsilon_0 A^2 L} r\, dr + \int_A^{5A} dr\, \frac{Q}{2\pi \varepsilon_0 L r}$$

$$\Delta V = \frac{Q}{4\pi \varepsilon_0 L} + \frac{Q}{2\pi \varepsilon_0 L} \ln 5$$

(3)



$$1: \quad V - V + R i_2 - R i_1 = 0$$

$$2: \quad V - R_3 i_3 - R i_2 = 0$$

$$A: \quad i_1 + i_2 - i_3 = 0$$

$$1: \Rightarrow \quad i_1 = i_2$$

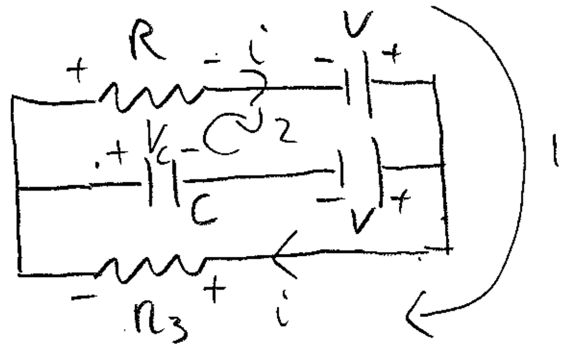
$$A: \Rightarrow \quad i_3 = 2 i_1$$

$$2: \Rightarrow \quad V = R_3 2 i_1 + R i_1 = (R + 2R_3) i_1$$

$$\Rightarrow \quad i_1 = \frac{V}{R + 2R_3} = i_2$$

$$i_3 = 2 i_1 = \frac{2V}{R + 2R_3}$$

(b)



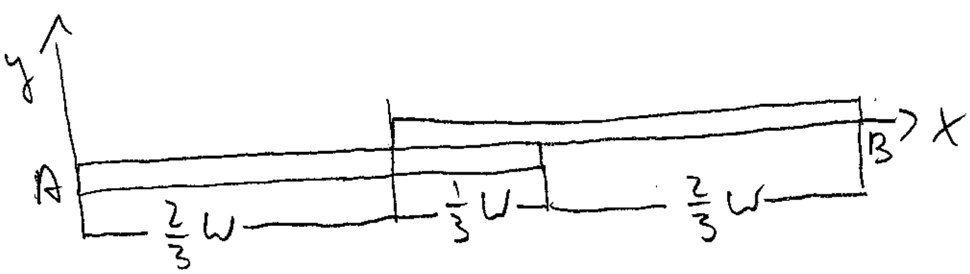
$$1:\quad V - (R + R_3)\, i = 0 \Rightarrow i = \frac{V}{R + R_3}$$

$$2:\quad V_C - R i = 0$$

$$\Rightarrow V_C = R i = \frac{R}{R + R_3} V$$

$$Q = C V_c = C \frac{R}{R + R_3} V$$

(4)



(a) $\vec{j} = j(x)\,\vec{i}_x$ with

$$j(x) = \begin{cases} \dfrac{i}{A} & \text{for } 0 \le x \le < w \\ \dfrac{i}{2A} & \text{for } \frac{2}{3}w < x < w \\ \dfrac{i}{A} & \text{for } w < x < \frac{5}{3}w \end{cases}$$

(b) It's a series circuit of 3 resistors with Resistances

$$R_1 = \frac{8\frac{2}{3}w}{A} = \frac{28w}{3A}$$

$$R_2 = \frac{8\frac{1}{3}w}{2A} = \frac{8w}{6A}$$

$$R_3 = \frac{28w}{3A}$$

$$V_A - V_B = (R_1 + R_2 + R_3)\,i = \frac{8wi}{A}\left(\frac{2}{3} + \frac{1}{6} + \frac{2}{3}\right) = \frac{38wi}{2A}$$

This can also be obtained from $\vec{E} = 8\vec{j}$ and integration

(c) $R_{total} = R_1 + R_2 + R_3 = \dfrac{V_A - V_B}{i} = \dfrac{38w}{2A}$