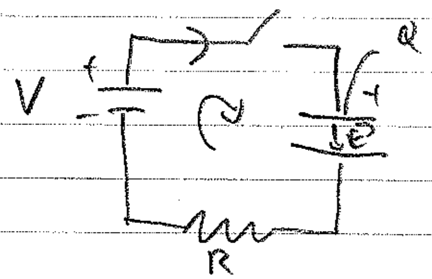
# 13 Maxwell's equations and electromagnetic waves

## 13.1 The displacement current

We look again at the RC circuit




We used Faraday's law to find the current as function of time:

$$\oint \vec{E}\, d\vec{r} = -V + \frac{Q}{C} + iR = 0$$

where we have neglected the self-inductance of the loop. Further we have

$$i = \frac{dQ}{dt}$$

and thus

$$R\frac{dQ}{dt} + \frac{Q}{C} = V$$

The solution is

$$Q = CV + A\exp\left(-\frac{t}{RC}\right)$$

With $Q=0$ at $t=0$

$$Q = CV\left[1-\exp\left(-\frac{t}{RC}\right)\right]$$

and $$i = \frac{dQ}{dt} = \frac{V}{R}\exp\left(-\frac{t}{RC}\right)$$

According to Ampère's circuital law the current leads to a $\vec{B}$ field, where

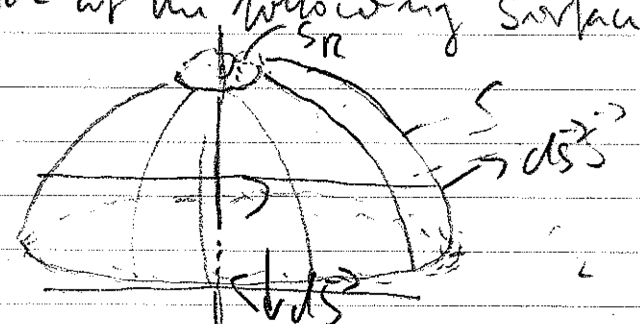
$$\int_{\partial S} d\vec{r}\, \vec{B} = \mu_0 i$$

where $\partial S$ is the boundary of an arbitrary surface through which the current $i$ goes. Now we look at the upper plate of the capacitor


Let $S$ be a disk of radius $r$ with the circle oriented such that $d\vec{S}$ is going in direction of the current. Then

$$\oint_{\partial S_r} d\vec{r} \cdot \vec{B} = B\, 2\pi r = \mu_0 i$$

Now we look at the following surface




This dome-like surface also has $\partial S_R$ as a boundary but on the other hand there is no current running through it. This leads to the contradiction

$$\int_S d\vec{S}\, \vec{j} = 0 \quad \text{but} \quad \int_{\partial S} d\vec{r}\, \vec{B} = \mu_0 i$$

Maxwell's idea to solve the problem with Ampère's law was the observation that there is a time-changing electric field $\vec{E}$ between the plates of the capacitor which is given by Gauss's law

$$\vec{E} = \frac{Q}{\varepsilon_0 A}(-\vec{i}_y)$$

Thus the flux of $\vec{E}$ through $S$ is

$$\Phi_{\vec{E}} = \int_S d\vec{S}\cdot\vec{E} = \frac{Q}{\varepsilon_0}$$

when only the plane bottom surface inside the capacitor contributes.
Now on the other hand we have

$$i = \frac{dQ}{dt} = \varepsilon_0 \frac{d\Phi_{\vec{E}}}{dt}$$

Thus we can save Ampère's law by adding $\frac{d\Phi_E}{dt}\varepsilon_0$ to the current. That's called the displacement current.

We can also write it in the local form

$$\oint_{\partial S} d\vec{r}\,\vec{B} = \mu_0 \int_S d\vec{S}\left(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right),$$

where $\vec{j}$ is the current density, and the orientations $S$ and $\partial S$ have to be related by the right-hand rule.

Also the current conservation law is fulfilled, if we

Consider the displacement current in the law. Take now the complete hemisphere $S'$, enclosing the capacitor plate

$$\oint_{S'} d\vec{S}\,(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}) = -i + \varepsilon_0 \frac{d\Phi_{\vec{E}}}{dt} = 0,$$

because according to Maxwell's law

$$\oint_{S'} d\vec{S}\,\varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \int_{\text{bottom}} d\vec{S}\,\varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \varepsilon_0 \frac{d}{dt} \int_{\text{bottom}} d\vec{S}\cdot\vec{E}$$

$$\overset{\text{Gauss's Law}}{=} \frac{dQ}{dt} = i$$

In other words

$$\oint_{S'} d\vec{S}\,\vec{j} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V dV\,\rho$$

where $\partial V = S'$ and the orientation of $\partial V$ is always out of the volume.

## 13.2 The complete set of Maxwell's Equations

We can now write all our knowledge about electromagnetism in terms of four equations

(I) $\oint_{\partial V} d\vec{S}\cdot\vec{E} = \frac{1}{\varepsilon_0}\int_V dV\rho$ Gauss's law $\Leftrightarrow$ electrical charges are the source of an electric field

(II) $\oint_{\partial V} d\vec{S}\cdot\vec{B} = 0$ No magnetic charges ("monopoles") exist.

(III) $$\oint_{\partial S} d\vec{r}\,\vec{E} = -\frac{d}{dt}\int_S d\vec{S}\cdot\vec{B}$$ (Faraday's law)

(IV) $$\oint_{\partial S} d\vec{r}\,\vec{B} = \mu_0 \int_S d\vec{S}\left(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$ Ampère's law + Maxwell's Displacement current

Here always

- Boundaries $\partial V$ of a volume $V$ are closed surfaces with the normal vectors pointing out of the volume.

- The direction of the boundaries, $\partial S$, of a surface, $S$, (which are closed lines) determine the direction of the normal vectors of this surface, $S$, according to the right-hand rule:
  Curving the fingers of the right hand in direction of the line, $\partial S$, the thumb points in direction of $d\vec{S}$!

## 13.3. Current conservation again

If we use a closed surface $S$ in (IV), the left-hand side is 0, because a closed surface has no boundary:

$$\oint_{S=\partial V} d\vec{S}\left(\vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right) = 0$$

$\partial V$ is the boundary of the volume enclosed by the surface $S$.

If the surface is further fixed in time (not moving in any way), then we can write

$$\oint_{\partial V} d\vec{S}\,\frac{\partial \vec{E}}{\partial t} = \frac{d}{dt}\oint_{\partial V} d\vec{S}\,\vec{E}$$

$$\overset{(I)}{=} \frac{1}{\varepsilon_0}\frac{d}{dt}\int_V dV\,\rho(t,\vec{r})$$

Thus:

$$\oint_{\partial V} d\vec{S}\cdot\vec{j}(t,\vec{r}) = -\frac{d}{dt}\int_V dV\,\rho(t,\vec{r})$$

Any charge flowing out of the boundary of a volume must come from the inside of this volume. The total number of charges must be conserved.

## 13.4. Electromagnetic waves

The most important result from Maxwell's completion of the laws of Electricity and Magnetism is his prediction of electromagnetic waves, i.e., electromagnetic fields which propagate through space without currents and charges!

For this case Maxwell's equations read

$$\oint_{\partial V} d\vec{S}\,\vec{E} = 0 \quad \text{(no charges)}$$

$$\oint_{\partial V} d\vec{S}\,\vec{B} = 0 \quad \text{(no magnetic monopoles)}$$

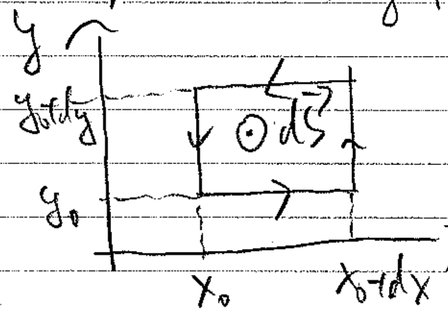
$$\oint_{\partial S} d\vec{r}\,\vec{E} = -\frac{d}{dt}\int_S d\vec{S}\cdot\vec{B}$$

$$\oint_{\partial S} d\vec{r}\,\vec{B} = \mu_0 \varepsilon_0 \frac{d}{dt}\int_S d\vec{S}\,\vec{E}$$

Now we like to find the most simple set of fields, $\vec{E}$ and $\vec{B}$, to obey this set of equations. We assume that

$$\vec{E} = E_y(t,x)\,\vec{i}_y$$

We start with Faraday's law for a infinitesimally small rectangle in the $xy$ plane:




Since $E_x = 0$, there is no contribution from the horizontal parts. So we find

$$\oint_{\partial S} d\vec{r}\cdot\vec{E} = \left[E_y(t,x_0+dx) - E_y(t,x_0)\right]dy$$

$$\overset{!}{=} -\frac{\partial}{\partial t} B_z(t,x_0)\,dx\,dy$$

$$\Rightarrow \frac{E_y(t,x_0+dx) - E_y(t,x_0)}{dx} = -\frac{\partial B_z(t,x_0)}{\partial t}$$

For $dx \to 0$ we find

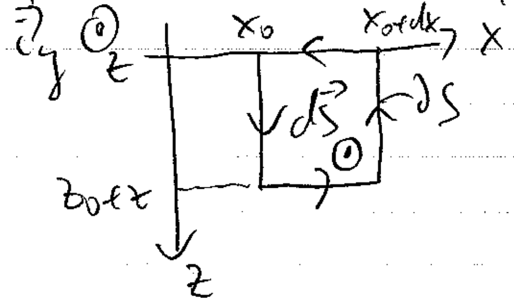
$$\boxed{\frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}} \qquad (M)$$

we assume that

$$\vec{B} = B_z(t,x)\,\vec{i}_z$$



In Ampère-Maxwell's law, we use the path




$$-\oint d\vec{r}\,\vec{B} = \left[B_z(t,x_0) - B_z(t,x_0+dx)\right] dz$$

$$= \mu_0 \varepsilon_0 \frac{\partial E_y(t,x_0)}{\partial t}\,dx\,dz$$

$$\overset{dz\to 0}{\Longrightarrow} \quad \boxed{-\frac{\partial B_z}{\partial x} = \mu_0 \varepsilon_0 \frac{\partial E_y}{\partial t} \qquad (B)}$$

Differentiating (B) w.r.t. $t$ we get

$$-\frac{\partial^2 B_z}{\partial t\,\partial x} = \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \qquad (C)$$

and (B) wrt $x$:

$$\frac{\partial^2 E_y}{\partial x^2} = -\frac{\partial^2 B_z}{\partial x\,\partial t} = -\frac{\partial^2 B_z}{\partial t\,\partial x} \qquad (D)$$

Plugging this into (C) we find

$$\frac{1}{c^2}\frac{\partial^2 E_y}{\partial t^2} = \frac{\partial^2 E_y}{\partial x^2} \quad \text{with } c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$$

This is the wave equation. It is easy to see that for any function, $f$,

$$E_y(t,x) = f(kx - \omega t)$$

is a solution, we only have to determine a relation for the constants $\omega$ and $k$:

$$\frac{\partial E_y}{\partial t} = -\omega f'(kx - \omega t) \; ; \quad \frac{\partial E_y}{\partial x} = k f'(kx - \omega t)$$

$$\frac{\partial^2 E_y}{\partial t^2} = \omega^2 f''(kx - \omega t) \; ; \quad \frac{\partial^2 E_y}{\partial x^2} = k^2 f''(kx - \omega t)$$

Plugging this into the wave equation gives you

$$\left(\frac{\omega^2}{c^2} - k^2\right) f''(kx - \omega t) = 0$$

which is solved by relating $\omega$ and $k$ to make the bracket vanish

$$k^2 = \frac{\omega^2}{c^2} \quad \text{or} \quad k = \pm \frac{\omega}{c}$$

Thus we find even a more general solution, namely

$$E_y(t,x) = f_1(kx - \omega t) + f_2(-kx - \omega t)$$

which describes the superposition of waves running in the positive ($k = +\frac{\omega}{c}$) and negative ($k = -\frac{\omega}{c}$) x direction (provided $\omega > 0$).

Particularly important are waves with a fixed oscillation frequency. For a wave running in +x direction this is

$$E_y(t,x) = A \cos(kx - \omega t) \quad \text{with} \quad k = +\frac{\omega}{c} > 0$$

or
$$E_y(t,x) = A\cos\left[\omega\left(\frac{x}{c} - t\right)\right]$$

The period is determined by

$$\omega T = 2\pi \Rightarrow \omega = \frac{2\pi}{T} = 2\pi f$$

and in space by

$$k\lambda = 2\pi \Rightarrow k = \frac{2\pi}{\lambda} \quad (\lambda\text{: wave length})$$

from $k = \frac{\omega}{c}$ we find

$$\frac{2\pi}{\lambda} = \frac{2\pi}{cT} \Rightarrow \boxed{\lambda = cT}$$

The phase of the wave thus moves with the velocity, c.

This has been determined to be

$$c = \sqrt{\frac{1}{\mu_0 \varepsilon_0}}$$

$$\left.\begin{aligned} \text{Now } \varepsilon_0 &= \frac{1}{4\pi \cdot 9\cdot 10^9} \frac{C^2}{Nm^2} \\ \mu_0 &= 4\pi \cdot 10^{-7} \frac{N\sec^2}{C^2} \end{aligned}\right\} \Rightarrow c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3\cdot 10^8 \frac{m}{s}$$

c is the speed of light and led Maxwell to the conclusion that light is an electromagnetic wave.

The magnetic field is now determined by Eq. (A)

$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} = A k \sin(kx - \omega t)$$

Up to a constant B-field thus

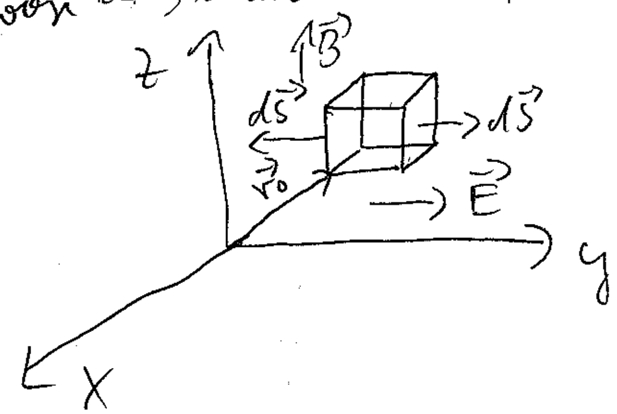
$$B_z = \frac{Ak}{\omega} \cos(kx - \omega t)$$

$$B_z = \frac{A}{c} \cos(kx - \omega t)$$

The $\vec{B}$-field is thus perpendicular to $\vec{E}$. Both $\vec{E}$ and $\vec{B}$ are perpendicular to the wave's propagation direction.
Finally we have to check that our solution also fulfills the remaining Maxwell equations for the case that there are no charges present, i.e.

$$\oint_{\partial V} d\vec{S} \cdot \vec{E} = 0 \quad \text{and} \quad \oint_{\partial V} d\vec{S} \cdot \vec{B} = 0$$

For $V$ we choose a small box at $\vec{r}_0$:




$$\oint_{\partial V} d\vec{S} \cdot \vec{E} = [E_y(t, \vec{r}_0 + dy\,\vec{i}_y) - E_y(t, \vec{r}_0)]\, dx\, dz$$

$$= 0,$$

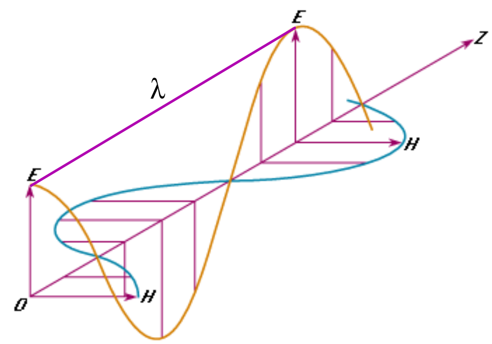
because $E_y$ does not depend on $y$, only on $x$.
For the $\vec{B}$ field we have

$$\oint_{\partial V} d\vec{S} \cdot \vec{B} = [B_z(t, \vec{r}_0 + dz\,\vec{i}_z) - B_z(t, \vec{r}_0)]\, dx\, dy = 0$$

since $\vec{B}$ does not depend on $z$.

## Characteristics of "free" em. Waves

- electric and magnetic fields oscillate $\perp$ to direction of propagation $\Rightarrow$ **transverse waves**
- $\vec{E} \perp \vec{B}$ are in phase
- phase velocity: $c = 1/\sqrt{\mu_0 \epsilon_0}$ = **speed of light**
- dispersion relation: $\omega = 2\pi f = c|\vec{k}|$ or $\lambda = cT$ ($|\vec{k}| = 2\pi/\lambda$.
- sources (not explained in this course)
  - accelerated charged particles =
  - **time-dependent** electric charges and and currents
  - modern picture: quantum-mechanical transitions in atoms (visible light, UV) and nuclei ($\gamma$ rays)






## The em. spectrum

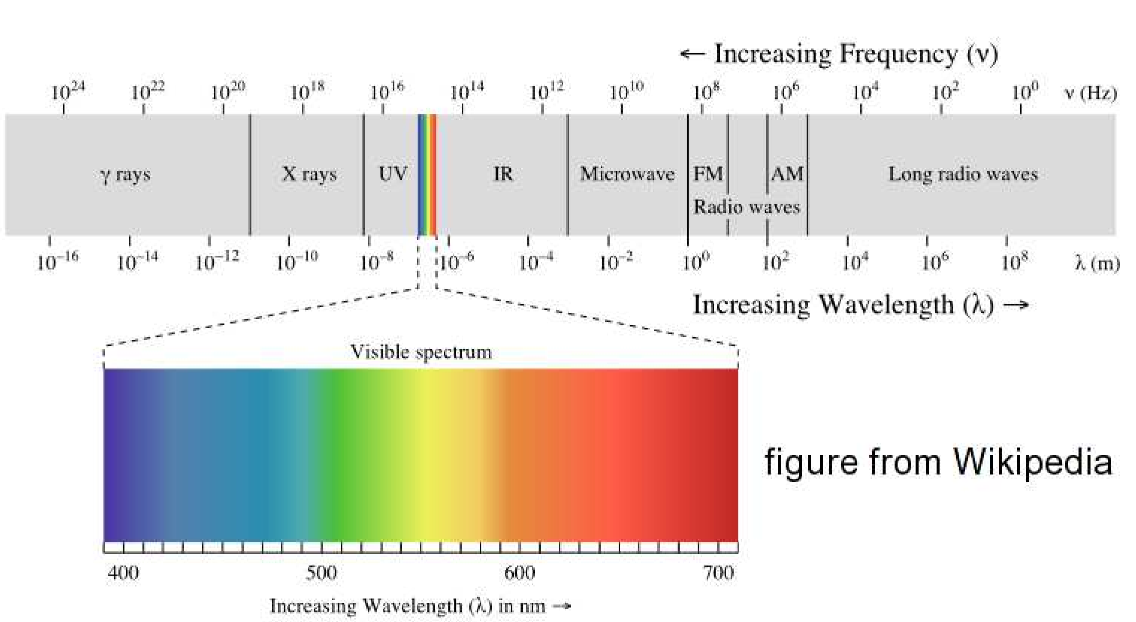



figure from Wikipedia

# The em. spectrum

| | $f$ (Hz) | $\lambda$ (m) | source (ex.) |
|---|---|---|---|
| $\gamma$ rays | $> 10^{20}$ | $< 10^{-12}$ | radioactivity, nuclear transitions |
| X rays | $> 3 \cdot 10^{16}$ | $< 10^{-8}$ | bremsstrahlung radiation, atomic transitions |
| UV | $7.5 \cdot 10^{14}$-$3 \cdot 10^{16}$ | $4 \cdot 10^{-7}$ - $10^{-8}$ | our Sun |
| visible light | $4 \cdot 10^{14}$-$7.5 \cdot 10^{14}$ | $4 \cdot 10^{-7}$ - $7.5 \cdot 10^{-7}$ | atomic transitions |
| infrared | $3 \cdot 10^{11}$ - $4 \cdot 10^{14}$ | $10^{-3}$-$7.5 \cdot 10^{-7}$ | transitions between vibrational modes of molecules |



# The em. spectrum

| | $f$ (Hz) | $\lambda$ (m) | source (ex.) |
|---|---|---|---|
| Millimeter Waves | $30$-$300 \cdot 10^{9}$ | $10^{-3}$-$10^{-2}$ | antenna |
| microwaves | $1.6$-$30 \cdot 10^{9}$ | $10$-$187 \cdot 10^{-3}$ | magnetrons (microwave oven), rotation and torsion transitions of molecules, cosmic microwave backgrond |
| radio waves | $5 \cdot 10^{5}$-$1.6 \cdot 10^{9}$ | $0.2$-$200$ | antenna |