

## Physics 208 Quiz 3 Solutions

February 06, 2008 (due: February 20, 2008)

### Problem 1 (40 points)

Check for each of the following forces, whether they are conservative! If so, determine the corresponding potentials! In the following  $A$  is considered a constant.

(a)  $\vec{F}(\vec{r}) = A\vec{r}$ .

If the force is conservative, then there is a potential with  $\vec{F} = -\nabla U$ . Up to an arbitrary constant this gives

$$\begin{aligned} F_x = -\frac{\partial U}{\partial x} = Ax &\Rightarrow U = -\frac{A}{2}x^2 + U_1(y, z), \\ F_y = -\frac{\partial U}{\partial y} = Ay &\Rightarrow U = -\frac{A}{2}(x^2 + y^2) + U_2(z), \\ F_z = -\frac{\partial U}{\partial z} = Az &\Rightarrow U = -\frac{A}{2}(x^2 + y^2 + z^2) = -\frac{A}{2}r^2. \end{aligned} \quad (1)$$

$\vec{F}$  is conservative, and the potential is as given above.

(b)  $\vec{F}(\vec{r}) = Axz\vec{i}_x$ . (Here,  $\vec{i}_x$  is the unit vector of a Cartesian coordinate system  $\{\vec{i}_x, \vec{i}_y, \vec{i}_z\}$  as usual!)

If  $\vec{F}$  is conservative, we must have  $F_y = -\partial U/\partial y = 0$  and  $F_z = -\partial U/\partial z = 0$ . This would imply that  $U = U(x)$ , but  $F_x = Axz$  is dependent on  $z$ . So  $\vec{F}$  is not conservative, and thus there doesn't exist a potential for it.

(c)  $\vec{F}(\vec{r}) = A\vec{r}/|\vec{r}|^4$ .

This is a radial force, and thus if it is conservative, we must have  $U = U(r)$ , because then with the chain rule<sup>1</sup>

$$\vec{F} = -\frac{dU(r)}{dr}\nabla r = -\frac{dU(r)}{dr}\frac{\vec{r}}{r} \Rightarrow \frac{dU(r)}{dr} = -\frac{A}{r^3} \Rightarrow U(r) = \frac{A}{2r^2}. \quad (2)$$

(d)  $\vec{F}(\vec{r}) = A(x\vec{i}_x + y\vec{i}_y)/(x^2 + y^2)$ .

Suppose  $\vec{F}$  is conservative. Then

$$F_x = -\frac{\partial U}{\partial x} = A\frac{x}{x^2 + y^2} \Rightarrow U(x, y) = -\frac{A}{2}\ln\left(\frac{x^2 + y^2}{\rho_0^2}\right) + U_1(y), \quad (3)$$

where  $\rho_0$  is an arbitrary constant with dimension length (note that you should never have a dimensionful quantity as an argument of a logarithm, exponential function, trigonometric functions, etc.).

Further we find:

$$F_y = -\frac{\partial U}{\partial y} = A\frac{y}{x^2 + y^2} + U_1'(y) \Rightarrow U_1 = 0. \Rightarrow U = -\frac{A}{2}\ln\left(\frac{r^2}{\rho_0^2}\right) = -A\ln\left(\frac{r}{\rho_0}\right). \quad (4)$$

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<sup>1</sup> $\nabla r = \nabla\sqrt{x^2 + y^2 + z^2} = x/\sqrt{x^2 + y^2 + z^2}\vec{i}_x + \dots = \vec{r}/r$

**Problem 2** (60 points)

A particle with positive charge,  $q$ , moves in an electric field with a potential

$$V(\vec{r}) = \frac{A}{2} r^2.$$

You can assume that the particle is fixed in the  $xy$  plane, i.e.,  $\vec{r} = x\vec{i}_x + y\vec{i}_y$ , where  $A = \text{const}$  and  $A > 0$ .

- (a) What is the electric field, given by the electric potential,  $V$ ?

$$\vec{E} = -\nabla V = -A\vec{r}. \quad (5)$$

- (b) What is the force exerted on the particle?

$$\vec{F} = q\vec{E} = -qA\vec{r}. \quad (6)$$

- (c) Write down the equations of motion for the particle.

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F} = -qA\vec{r}. \quad (7)$$

- (d) Solve the equations of motion!

**Hint:** Show that for the right  $\omega = \text{const}$  (which is it?)

$$\vec{r}(t) = \vec{c}_1 \cos(\omega t) + \vec{c}_2 \sin(\omega t) \quad (8)$$

is a solution for arbitrary constant vectors,  $\vec{c}_1$  and  $\vec{c}_2$ . Then check, whether you can find always  $\vec{c}_1, \vec{c}_2$  such that Eq. (8) is a solution for an arbitrarily given initial condition

$$\vec{r}(0) = \vec{r}_0, \quad \vec{v}(0) = \dot{\vec{r}}(0) = \vec{v}_0.$$

Take the 2<sup>nd</sup> derivative of the ansatz (8) with respect to time:

$$\frac{d^2 \vec{r}}{dt^2} = -\vec{c}_1 \omega^2 \cos(\omega t) - \vec{c}_2 \omega^2 \sin(\omega t) = -\omega^2 \vec{r} \quad (9)$$

This solves the equations of motion, if we set

$$\omega = \sqrt{\frac{qA}{m}}, \quad (10)$$

and then we can choose  $\vec{c}_1$  and  $\vec{c}_2$  to fulfill the initial conditions  $\vec{r}(t=0) = \vec{r}_0$  and  $\vec{v}(t=0) = \dot{\vec{r}}(t=0) = \vec{v}_0$  by setting  $\vec{c}_1 = \vec{r}_0$  and  $\vec{c}_2 = \vec{v}_0/\omega$ . The solution of the initial-value problem is thus

$$\vec{r}(t) = \vec{r}_0 \cos(\omega t) + \frac{\vec{v}_0}{\omega} \sin(\omega t), \quad (11)$$

where  $\omega$  is given by (10).

- (e) Determine the trajectory of a particle whose initial condition is given as follows:

$$\vec{r}_0 = x_0 \vec{i}_x, \quad \vec{v}_0 = v_0 \vec{i}_y. \quad (12)$$

**Hint:** Even if you cannot solve problem (d), you can use Eq. (8) to solve problem (e)!

In this case

$$\vec{r}(t) = r_0 \cos(\omega t) \vec{i}_x + \frac{v_0}{\omega} \sin(\omega t) \vec{i}_y. \quad (13)$$

The trajectory is an ellipse with semi axes  $r_0$  and  $v_0/\omega$  with the center in the origin of the  $xy$  plane.

(f) What is the energy of the particle, given the initial conditions (12)? Is this energy conserved?

With  $U = qV$ , the energy is given by the initial conditions:

$$E = \frac{m}{2}v_0^2 + \frac{qA}{2}r_0^2 = \frac{m}{2}(v_0^2 + \omega^2 r_0^2). \quad (14)$$

Since the force is conservative, we know that the total energy is conserved. Of course, one can verify this with help of the equations of motion by differentiation of the energy wrt. time:

$$E = \frac{m}{2}\dot{r}^2 + U(\vec{r}) \Rightarrow \frac{dE}{dt} = m\dot{r}\ddot{r} + \dot{\vec{r}}\nabla U(\vec{r}) = \dot{\vec{r}}(m\ddot{\vec{r}} - \vec{F}) = 0. \quad (15)$$