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# Physics 208 Quiz 8

April 11, 2008; due April 18, 2008

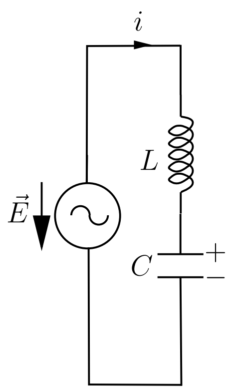
**Problem 1** (60 Points)

Consider the AC circuit consisting of an AC voltage in series with a coil of self-inductance, $L$, and a capacitor of capacitance, $C$. The voltage has the time dependence

$$V(t) = V_0 \cos(\omega_0 t). \tag{1}$$

Assume that

$$\omega_0 \neq \frac{1}{\sqrt{LC}}. \tag{2}$$




(a) Derive the differential equation for the charge on the upper plate (labeled + in the diagram) of the capacitor.

**Hint:** The solution for this part is

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + Q/C = V(t). \tag{3}$$

(b) Find the solution for $Q$ and $i = \frac{\mathrm{d}Q}{\mathrm{d}t}$ for the initial condition $Q(t = 0) = Q_0$, $i(t = 0) = 0$.

(c) Assume now that there is a small resistance, $R$[1], such that after a long time the "transient part" (i.e., the part of the solution which is given by the solution for the homogeneous equation) is damped out, and $i$ obeys the the "steady-state" solution

$$i(t) = \hat{\imath} \cos(\omega_0 t - \phi_0). \tag{4}$$

Calculate the amplitude for the current, $\hat{\imath}$, and the phase shift, $\phi_0$, as a function of $L$, $C$, $V_0$, and $\omega_0$. Show that this case is contained in the general solution for the damped $RLC$ circuit discussed in the lecture for $R \to 0$.

(d) [for extra credit] Work out the solutions for questions (a) and (b) in the "resonance case", i.e., for $\omega_0 \to \omega_R$, where $\omega_R = 1/\sqrt{LC}$.

**Hint:** To find a particular solution of the inhomogeneous equation, make the ansatz

$$Q_I(t) = q(t) \sin(\omega_0 t) \tag{5}$$

and solve for q.

[1] You do not need to give solutions for this more general case which is treated in detail in the lecture. You can look it up in the lecture notes on the course home page.

## Solutions

(1a) Using Faraday's Law, integrating along the wires in direction of the current, one gets

$$\oint \mathrm{d}\vec{r}\vec{E} = -V + \frac{Q}{C} = -\frac{\mathrm{d}}{\mathrm{d}t}(Li) = -L\frac{\mathrm{d}i}{\mathrm{d}t}, \qquad \text{(10 points)} \tag{6}$$

because here $L$ does not change with time. The current is related to the charge $Q$ on the upper plate of the capacitor by

$$i = \frac{\mathrm{d}Q}{\mathrm{d}t}, \qquad \text{(5 points)} \tag{7}$$

and thus we obtain from (6)

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + \frac{Q}{C} = V(t) = V_0\cos(\omega_0 t). \qquad \text{(5 points)} \tag{8}$$

(b) To solve the equation, we need first the general solution of the homogeneous equation,

$$\frac{\mathrm{d}^2Q_H}{\mathrm{d}t^2} = -\frac{Q_H}{LC}, \tag{9}$$

which is given by

$$Q_H(t) = A\cos(\omega t) + B\sin(\omega t) \text{ with } \omega = \frac{1}{\sqrt{LC}}. \qquad \text{(10 points)} \tag{10}$$

To find a particular solution of the inhomogeneous equation, we make the ansatz

$$Q_I(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t) \qquad \text{(10 points)} \tag{11}$$

Plugging this into Eq. (8) we find by comparison of coefficients in front of $\cos(\omega_0 t)$ and $\sin(\omega_0 t)$:

$$a = \frac{CV_0}{1 - LC\omega_0^2}, \quad b = 0. \qquad \text{(5 points)} \tag{12}$$

Note that it is important that we assumed $\omega_0 \neq 1/\sqrt{LC}$!
The complete solution is the sum of the homogeneous and the inhomogeneous solution:

$$Q(t) = \frac{CV_0}{1 - LC\omega_0^2}\cos(\omega_0 t) + A\cos(\omega t) + B\sin(\omega t). \tag{13}$$

The current is given by (7)

$$i(t) = \frac{\mathrm{d}Q(t)}{\mathrm{d}t} = -\frac{CV_0\omega_0}{1 - LC\omega_0^2} - A\omega\sin(\omega t) + B\omega\cos(\omega t). \qquad \text{(5 points)} \tag{14}$$

The constants $A$ and $B$ are determined by the initial conditions given in the question:

$$i(t=0) = B\omega = 0 \Rightarrow B = 0, \quad Q(t=0) = \frac{CV_0}{1 - LC\omega_0^2} + A = Q_0 \Rightarrow A = Q_0 - \frac{CV_0}{1 - LC\omega_0^2}. \tag{15}$$

(5 points)

The final answer thus reads

$$\begin{aligned} Q(t) &= \frac{CV_0}{1-LC\omega_0^2}\cos(\omega_0 t) + \left(Q_0 - \frac{CV_0}{1-LC\omega_0^2}\right)\cos(\omega t), \\ i(t) &= -\frac{CV_0\omega_0}{1-LC\omega_0^2}\sin(\omega_0 t) - \left(Q_0 - \frac{CV_0}{1-LC\omega_0^2}\right)\omega\sin(\omega t). \end{aligned} \tag{16}$$

(c) The steady-state solution for the current in the sense that we may assume that the transient solution is damped away after a long time due to a small resistance, reads

$$i(t) = -\frac{CV_0\omega_0}{1-LC\omega_0^2}\sin(\omega_0 t) = \frac{CV_0\omega_0}{1-LC\omega_0^2}\cos\left(\omega_0 t + \frac{\pi}{2}\right). \tag{17}$$

Thus the amplitude and phase shift are

$$\hat{i} = \frac{CV_0\omega_0}{1-LC\omega_0^2}, \quad \phi_0 = -\frac{\pi}{2} \qquad \text{(5 points)}. \tag{18}$$

These results one finds indeed also from the lecture notes p. 127-128 for $R = 0$. Especially the phase shift is $\phi_0 = \arccos(1) - \pi/2 = -\pi/2$.

(d) For $\omega_0 = \omega = 1/\sqrt{LC}$, one has to make the special ansatz, given in the problem:

$$Q_I(t) = q(t)\sin(\omega_0 t). \tag{19}$$

Plugging this in (8), one gets after comparison of coefficients in front of $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$ on both sides of the resulting equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = 0, \quad 2\omega_0\frac{\mathrm{d}q}{dt} = \frac{V_0}{L} \qquad \text{(+10 extra)} \tag{20}$$

with the special solution

$$q(t) = \frac{V_0}{2\omega_0 L}t. \qquad \text{(+5 extra)} \tag{21}$$

So the general solution also in this case is the sum of the general solution of the homogeneous equation and the just found special solution of the inhomogeneous equation:

$$Q(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{V_0}{2L\omega_0}t\sin(\omega_0 t). \qquad \text{(+5 extra)} \tag{22}$$

From the initial conditions one finds

$$A = Q_0, \quad B = 0 \Rightarrow Q(t) = Q_0\cos(\omega_0 t) + \frac{V_0}{2L\omega_0}t\sin(\omega_0 t), \qquad \text{(+5 extra)} \tag{23}$$

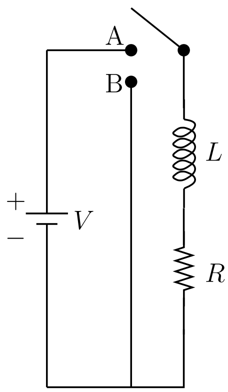
and the currents reads

$$i(t) = \frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{V_0}{2L\omega_0}\left[\omega_0 t\cos(\omega_0 t) + \sin(\omega_0 t)\right] - Q_0\omega_0\sin(\omega_0 t). \qquad \text{(+5 extra)} \tag{24}$$

The first term makes the amplitude of the current grow indefinitely. That's known as the "resonance catastrophe". Of course, in practice this cannot happen, because there is always a finite resistance, and there is no generator which can produce an indefinite power!

**Problem 2** (40 Points)

Consider the following circuit with a coil of self-inductance, $L$, and a resistor with resistance, $R$ (which is assumed to account also for the resistance of the coil).




(a) Assume that the switch has been set to position A for a very long time so that a constant current, $i_0$, runs through coil and resistor. What is this current, $i_0$?

(b) At $t = 0$ the switch is brought in position B. Derive the differential equation for the current through the resistor and solve for $i(t)$, assuming the appropriate initial condition.

(c) What is the power (energy per time) dissipated into heat in the resistor as a function of time (for $t \geq 0$)?

(d) What is the total energy dissipated into heat?

**Hint:** Integrate your result for the power from part (c) with respect to time from $t = 0$ to $t \to \infty$!

(e) What has been the total energy content of the coil immediately before switching to position B? Explain briefly, in which form this energy has been "stored" within the coil!

## Solutions

(a) If the switch has been closed in position A for a very long time, we can assume a time-independent current which is given by Ohm's Law:

$$i_0 = \frac{V}{R}. \qquad \text{(5 points)} \tag{25}$$

(b) For the switch in position B we need to evaluate Faraday's Law. Letting the current run from the top to the bottom through the resistor and the coil and using this as integration direction in Faraday's Law gives

$$\oint \mathrm{d}\vec{r}\vec{E} = Ri = -L\frac{\mathrm{d}i}{\mathrm{d}t}. \qquad \text{(5 points)} \tag{26}$$

This differential equation has the general solution

$$i(t) = A\exp\left(-\frac{R}{L}t\right). \qquad \text{(5 points)} \tag{27}$$

The initial condition is $i(t = 0) = A = i_0$ and thus

$$i(t) = i_0\exp\left(-\frac{R}{L}t\right). \qquad \text{(5 points)} \tag{28}$$

(c) The power dissipated into heat in the resistor is

$$P(t) = R[i(t)]^2 = R\,i_0^2\exp\left(-\frac{2R}{L}t\right). \qquad \text{(5 points)} \tag{29}$$

(d) The total energy dissipated into heat from the moment of switching the switch into position B is

$$E_{\text{tot}} = \int_0^\infty \mathrm{d}tP(t) = -Ri_0^2\,\frac{L}{2R}\exp\left(-\frac{2R}{L}t\right)\Bigg|_{t=0}^{t=\infty} = \frac{L}{2}i_0^2. \qquad \text{(10 points)} \tag{30}$$

(e) This means that immediately before we switched to position B, in the coil must have been stored this amount of energy, $E_{\text{tot}} = L/2i_0^2$, where $i_0$ is the current running through the coil at this moment. This energy has originally come from the battery building up a magnetic field inside the coil when the switch has been set to position A for a long time. The energy is thus stored in form of this magnetic field inside the coil! (5 points)

Total: 100 + 30 extra credit