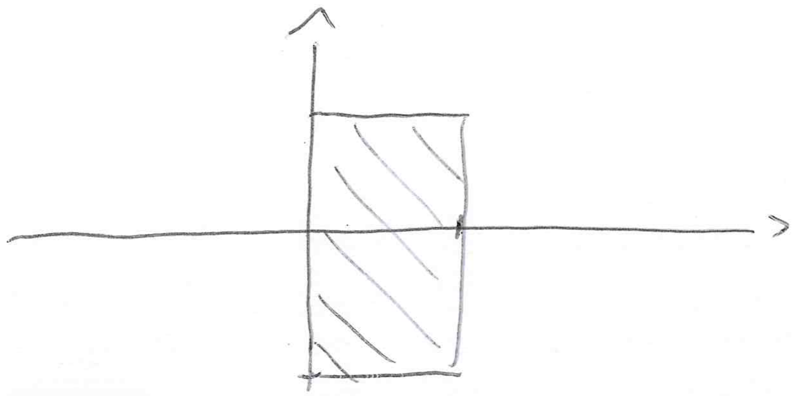
Integrals in 2D

$$\int_D f(x,y)\,dx\,dy$$

As a recall we do a simple example:

$$I = \int_D f(x,y)\,dx\,dy \quad \text{with} \quad D = \{ (x,y) \,/\, 0 < x < 1 \text{ and } y^2 < 1 \}$$

$$\text{and} \quad f(x,y) = x e^{xy}$$

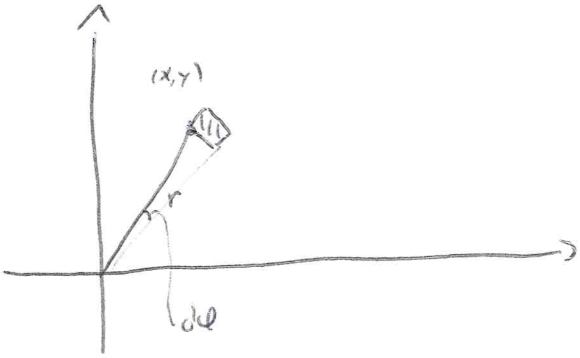


$$I = \int_0^1 dx \int_{-1}^{1} x\, e^{xy}\, dy = \int_0^1 dx \left[x \cdot \frac{1}{x} e^{xy} \right]_{-1}^{1} =$$

$$= \int_0^1 dx \left(e^{x} - e^{-x} \right) = \left(e^{x} + e^{-x} \right)_0^1 = e + e^{-1} - 1 - 1 =$$

$$= e + \frac{1}{e} - 2 = 1.08$$

We have then studied the transformation to polar coordinates...

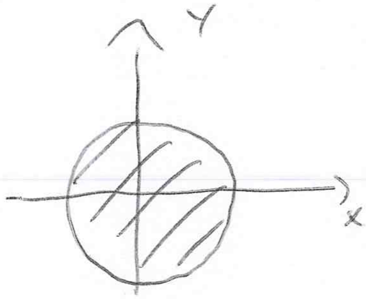


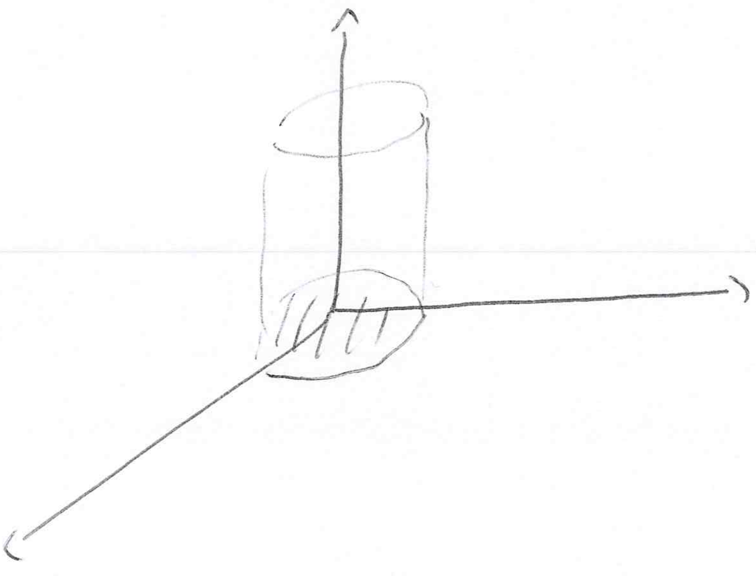
$dx\,dy = r\,dr\,d\varphi$

Example:

$$\int_D f(x,y)\,dx\,dy$$

with $f(x,y) = h > 0$ and $D: \{(x,y) / x^2+y^2 \leq R^2\}$.





$V = \pi R^2 \cdot h$

Let us transform:

$$D = \{ (r, \varphi) \;/\; 0 \le r \le R,\; 0 \le \varphi \le 2\pi \}$$

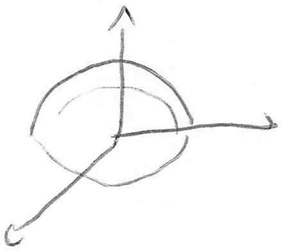
$$\int_D dx\,dy\; h \equiv \int_D r\,dr\,d\varphi \cdot h = \int_0^{2\pi} d\varphi \int_0^R r\,dr \cdot h =$$

$$= 2\pi \cdot \left[h \frac{r^2}{2} \right]_0^R = h\, 2\pi \frac{R^2}{2} = \pi R^2 \cdot h \;!!!$$

q.e.d.

$\equiv$

As an exercise there is the calculation of the volume of a half-sphere:



$$f(x,y) = \sqrt{R^2 - (x^2+y^2)}$$

$$D = \{ (x,y) \;/\; x^2+y^2 < R^2 \}.$$

$$\boxed{V = \frac{1}{2} \cdot \left(\frac{4}{3} \pi R^3 \right) = \frac{2}{3} \pi R^3}$$

In general, if we have the transformation of coordinates

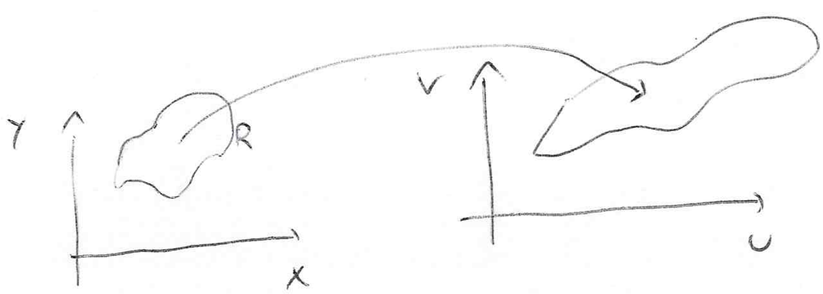
$$(x,y) \mapsto (u,v)$$

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

the following formula holds:

$$\int_R dx\,dy\, f(x,y) = \int_{R'} du\,dv\, f(x(u,v), y(u,v))\, J$$

whereas



$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

For instance, for $(u,v) = (r,\varphi)$:

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

$$J = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$$

$$dx\,dy \mapsto r\,dr\,d\varphi.$$

Note, this is valid also in more dimensions:

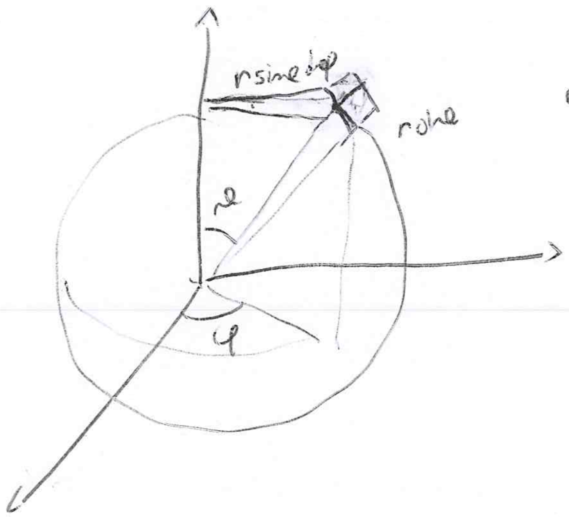
$$\int_R dx_1 \dots dx_N \, f(x_1, \dots, x_N)$$

$$x_i = x_i(y_1, \dots, y_N)$$

$$dx_1 \dots dx_N \to J \, dy_1 \dots dy_N$$

where
$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \frac{\partial x_1}{\partial y_N} \\ \dots & & \\ \frac{\partial x_N}{\partial y_1} & & \frac{\partial x_N}{\partial y_N} \end{vmatrix}$$

It is easy to prove that in spherical 3D coordinates one has



$$dV = r\,dr\,d\varphi \sin\theta\, d\theta\, r = r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

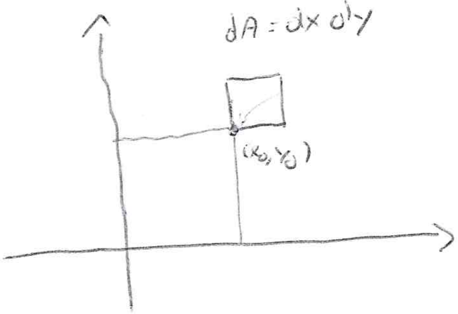
$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r \cos\theta \end{cases}$$

The check with the Jacobian is left as an ex.

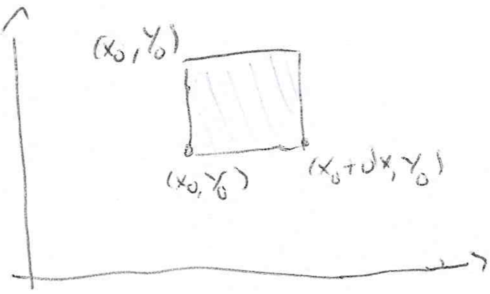
Let us consider a general transformation

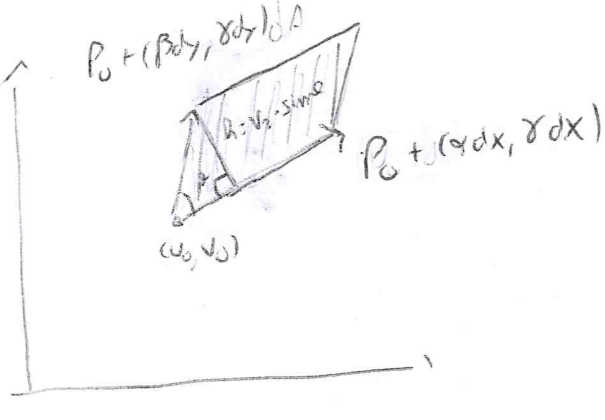
$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases} \qquad \begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

How do we transform $\int_R dx\,dy\, f(x,y)$?



Let us consider (x_0, y_0).





$(x_0, y_0) \longmapsto (u_0, v_0)$

$$\begin{cases} u(x,y) = u_0 + \left(\frac{\partial u}{\partial x}\right)_{\vec{x}_0}(x-x_0) + \left(\frac{\partial u}{\partial y}\right)_{\vec{x}_0}(y-y_0) = u_0 + \alpha(x-x_0) + \beta(y-y_0 \\ v(x,y) = v_0 + \left(\frac{\partial v}{\partial x}\right)_{\vec{x}_0}(x-x_0) + \left(\frac{\partial v}{\partial y}\right)_{\vec{x}_0}(y-y_0) = v_0 + \gamma(x-x_0) + \delta(y- \end{cases}$$

$(x_0 + dx, y_0) \mapsto (u, v) = (u_0, v_0) + (\alpha dx, \gamma dx)$

$(x_0, y_0 + dy) \mapsto (u, v) = (u_0, v_0) + (\beta dy, \delta dy)$

$(x_0 + dx, y_0 + dy) \mapsto (u, v) = (u_0, v_0) + (\alpha dx + \beta dy, \gamma dx + \delta dy)$

Which is the area?

$dA = |\vec{V_1}| \cdot |\vec{V_2}| \cdot \sin\vartheta$

$V_1 \circ V_2 = (\alpha dx, \gamma dx) \circ (\beta dy, \delta dy) = \alpha\beta\, dx\, dy + \gamma\delta\, dx\, dy$

$= \sqrt{\alpha^2 + \gamma^2}\, dx \sqrt{\beta^2 + \delta^2}\, dy \cdot \cos\vartheta$

Ergo:

$$\cos\vartheta = \frac{\alpha\beta + \gamma\delta}{\sqrt{\alpha^2+\gamma^2}\sqrt{\beta^2+\delta^2}}$$

$$\sin\vartheta = \sqrt{1 - \cos^2\vartheta} = \sqrt{1 - \frac{(\alpha\beta + \gamma\delta)^2}{(\alpha^2+\gamma^2)(\beta^2+\delta^2)}} =$$

$$= \sqrt{\frac{(\alpha^2+\gamma^2)(\beta^2+\delta^2) - (\alpha\beta + \gamma\delta)^2}{(\alpha^2+\gamma^2)(\beta^2+\delta^2)}} =$$

γ

$$= \sqrt{\frac{\cancel{\alpha^2\beta^2} + \alpha^2\delta^2 + \gamma^2\beta^2 + \cancel{\gamma^2\delta^2} - \cancel{\alpha^2\beta^2} - \cancel{\gamma^2\delta^2} - 2\alpha\beta\gamma\delta}{(\alpha^2+\gamma^2)(\beta^2+\delta^2)}}$$

$$= \sqrt{\frac{(\alpha\delta - \gamma\beta)^2}{(\alpha^2+\gamma^2)(\beta^2+\delta^2)}} = \frac{\alpha\delta - \gamma\beta}{\sqrt{\alpha^2+\gamma^2}\,\sqrt{\beta^2+\delta^2}}$$

δ_{90}:

$$dA = |\vec{V}_1|\cdot|\vec{V}_2|\cdot\sin\theta = (\alpha\delta - \gamma\beta)\,dx\,dy$$

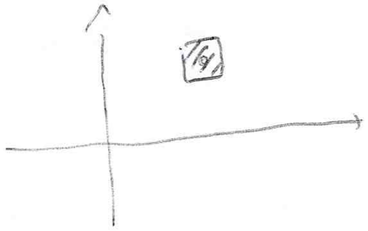
$=$

Now, one has that for a small integral in the rectangular $a \cdot b$

$$\int_D f(x,y)\,dx\,dy = f(x_0,y_0)\,a\cdot b$$

for a and b very small.

$$f(x_0,y_0) = h$$



Now, if we simply consider the integral in $du\,dv$ we get:

$$\int_{\tilde{D}} du\,dv\, f(u,v) = h\left(a \cdot b\ (\alpha\delta - \gamma\beta)\right)$$

$D \mapsto \tilde{D}$

That is, this is <u>not</u> equal... However, if we study:

$$\int_{\tilde{D}} du\,dv\, \frac{f(u,v)}{\alpha\delta - \gamma\beta} = \frac{h}{\cancel{\alpha\delta - \gamma\beta}}\left(a \cdot b\ (\cancel{\alpha\delta - \gamma\beta})\right)$$

Formally, we can consider the "inverse transformation"

$$\begin{cases} x - x_0 = \tilde{\alpha}(u - u_0) + \tilde{\beta}(v - v_0) \\ y - y_0 = \tilde{\gamma}(u - u_0) + \tilde{\delta}(v - v_0) \end{cases}$$

$\tilde{\alpha} = \left(\frac{\partial x}{\partial u}\right)_{u_0,v_0}$, $\tilde{\beta} = \left(\frac{\partial x}{\partial v}\right)$

$\tilde{\gamma} = \left(\frac{\partial y}{\partial u}\right)_{u_0,v_0}$, $\tilde{\delta} = \left(\frac{\partial y}{\partial v}\right)$

$$\begin{cases} u - u_0 = \alpha(x - x_0) + \beta(y - y_0) \\ v - v_0 = \gamma(x - x_0) + \delta(y - y_0) \end{cases}$$

We can express $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}$ as function of $\alpha, \beta, \gamma, \delta \dots$

$$\frac{1}{\alpha\delta - \gamma\beta} = \tilde{\alpha}\tilde{\delta} - \tilde{\gamma}\tilde{\beta} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}_{u_0, v_0}$$

Ergo:

$$\int_D f(x,y)\,dx\,dy = \int_{\tilde{D}} f(x(u,v), y(u,v)) \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right| du\,dv$$

Proof that:

$$\frac{1}{\alpha\delta-\gamma\beta} = \tilde{\alpha}\tilde{\delta} - \tilde{\gamma}\tilde{\beta}$$

$$\begin{cases} U - U_0 = \alpha(x-x_0) + \beta(y-y_0) \\ V - V_0 = \gamma(x-x_0) + \delta(y-y_0) \end{cases}$$

multiply by δ and β resp.:

$$\begin{cases} \delta(U-U_0) = \alpha\delta(x-x_0) + \beta\delta(y-y_0) \\ \beta(V-V_0) = \gamma\beta(x-x_0) + \delta\beta(y-y_0) \end{cases}$$

$\Rightarrow$ subtract:

$$\delta(U-U_0) - \beta(V-V_0) = (\alpha\delta-\gamma\beta)(x-x_0)$$

$$\boxed{(x-x_0) = \frac{\delta}{\alpha\delta-\gamma\beta}(U-U_0) - \frac{\beta}{\alpha\delta-\gamma\beta}(V-V_0)}$$

Simmillarly, multiply the first by γ and the 2^{nd} by α:

$$\begin{cases} \gamma(U-U_0) = \gamma\alpha(x-x_0) + \gamma\beta(y-y_0) \\ \alpha(V-V_0) = \alpha\gamma(x-x_0) + \alpha\delta(y-y_0) \end{cases}$$

$\Downarrow$ subtract:

$$\gamma(U-U_0) - \alpha(V-V_0) = -(\alpha\delta-\gamma\beta)(y-y_0)$$

Ergo:

$$\boxed{Y - Y_0 = -\frac{\gamma}{\alpha\delta - \gamma\beta}(U - U_0) + \frac{\alpha}{\alpha\delta - \gamma\beta}(V - V_0)}$$

It follows that:

$$\tilde{\alpha} = \frac{\delta}{\alpha\delta - \gamma\beta} \qquad \tilde{\beta} = \frac{-\beta}{\alpha\delta - \gamma\beta}$$

$$\tilde{\gamma} = \frac{-\gamma}{\alpha\delta - \gamma\beta} \qquad \tilde{\delta} = \frac{\alpha}{\alpha\delta - \gamma\beta}$$

$$\tilde{\alpha}\tilde{\delta} - \tilde{\gamma}\tilde{\beta} = \frac{\delta\alpha - \gamma\beta}{(\alpha\delta - \gamma\beta)^2} = \frac{1}{\alpha\delta - \gamma\beta}$$

q.e.d.