Convergence of a simple integral:

$$\int_{x_0}^{\infty} f(x)\,dx$$

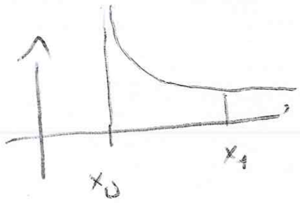
means that we integrate between "$x_0$" and "$\infty$". But care is needed, because this simble is well defined only as a limiting process.

Consider $F(x)$ / $\frac{dF(x)}{dx} = f(x)$. Then:

$$\int_{x_0}^{\infty} f(x)\,dx = \lim_{x \mapsto \infty} F(x) - F(x_0).$$

Similarly, if we consider the integral

$$\int_{x_0}^{x_1 > x_0} f(x)\,dx$$




where $f(x) : (x_0, \infty) \subseteq R \mapsto R$ (alias: $x_0$ is a point on the border of the domain of the function)

then:

$$\int_{x_0}^{x_1 > x_0} f(x)\,dx = F(x_1) - \lim_{x \mapsto x_0^+} F(x)$$

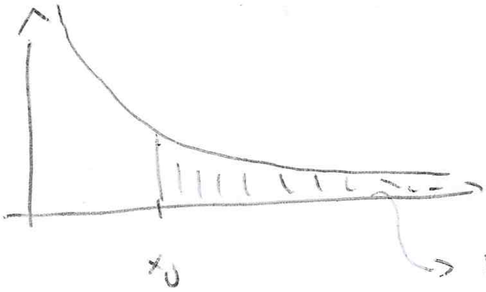
Let us study the "simplest" example:

$$f(x) = x^{\alpha} \qquad \text{for } \alpha > R .$$

$$\int_{x_0}^{\infty} x^{\alpha}\, dx . \qquad (x_0 > 0)$$

It is clear that for $\alpha > 0$ it diverges, because $f(x) = x^{\alpha > 0}$ grows and diverges for $x \mapsto \infty$.

But for $\alpha < 0$ we have something like



$\to$ Is this finite or not?

$$\int x^{\alpha}\, dx \overset{\alpha \neq 1}{=} \frac{x^{\alpha+1}}{\alpha+1}$$

$$\int_{x_0}^{\infty} x^{\alpha}\, dx = \left( \lim_{x \mapsto \infty} \frac{x^{\alpha+1}}{\alpha+1} \right) - \frac{x_0^{\alpha+1}}{\alpha+1} = \begin{cases} \infty & \text{for } \alpha+1 > 0 \Leftrightarrow \alpha > -1 \\ -\dfrac{x_0^{\alpha+1}}{\alpha+1} > 0 & \text{for } \alpha+1 < 0 \Leftrightarrow \alpha < -1 . \end{cases}$$

Ergo, $f(x) = \frac{1}{\sqrt{x}}$ is such that $\int_{x_0>0} \frac{1}{\sqrt{x}} \, dx$ diverges to $\infty$. ”

On the contrary, $f(x) = \frac{1}{x^2}$ is such that $\int_{x_0>0}^{\infty} \frac{1}{x^2} \, dx$ is convergent.

What about $\alpha = -1$?

$$\int_{x_0}^{\infty} \frac{1}{x} \, dx = \lim_{x \to \infty} \left( \ln \frac{x}{x_0} \right) = +\infty \quad ! \text{ it diverges} \ldots$$

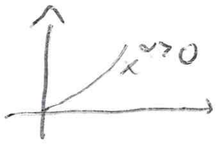
therefore:

$$\int_{x_0>0}^{\infty} x^{\alpha} \, dx \quad \begin{cases} = \infty & \text{for } \alpha \geq -1 \\ \text{finite} & \text{for } \alpha < -1 \end{cases}$$

Let us now consider the case

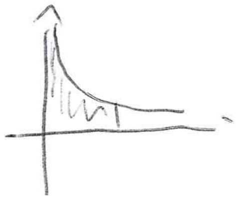
$$\int_0^{x_0>0} x^\alpha \, dx .$$

obviously, for $\alpha > 0$ the result is finite because $f(x=0)=0$ in this case and $[0, x_0]$ belongs to the domain of $f(x)$.




However, for $\alpha < 0$ one has

$$\lim_{x \mapsto 0^+} x^\alpha = +\infty$$



What about the integral?

$$\int x^\alpha dx \overset{\alpha \neq 1}{=} \frac{x^{\alpha+1}}{\alpha+1}$$

$$\int_0^{x_0} x^\alpha dx = \frac{x_0^{\alpha+1}}{\alpha+1} - \lim_{x \mapsto 0^+} \frac{x^{\alpha+1}}{\alpha+1} = \begin{cases} \dfrac{x_0^{\alpha+1}}{\alpha+1} \text{ (finite)} & \text{for } \alpha > -1 \\ +\infty & \text{for } \alpha < -1 \end{cases}$$

Thus, $f(x) = \frac{1}{\sqrt{x}}$ is such that $\int_0^{x_0>0} \frac{1}{\sqrt{x}} dx$ converges and

$f(x) = \frac{1}{x^2}$ is such that $\int_0^{x_0} \frac{1}{x^2} dx$ diverges.

(The opposite case when compared to $\int_{x_0}^{\infty} x^\alpha dx \ldots$)

For $\alpha = -1$

$$\int_0^{x_0} \frac{1}{x}\, dx = \lim_{x \mapsto 0^+} \ln \frac{x_0}{x} = +\infty$$

Then:

$$\int_0^{x_0} x^{\alpha}\, dx = \begin{cases} \text{finite} & \text{for } \alpha > -1 \\ \text{infinite} & \text{for } \alpha \leq -1 \end{cases}$$

# Asymptotic behavior of function and convergence / divergence of integrals

Consider the function $f(x): \mathbb{R} \mapsto \mathbb{R}$. We say that $f(x)$ for $x \mapsto \infty$ behaves as $x^\alpha$ if

$$\lim_{x \mapsto \infty} \frac{f(x)}{x^\alpha} = \text{const.}$$

For instance, $f(x) = (3x+1)^2$ behaves as $x^2$ for $x \mapsto +\infty$:

$$\lim_{x \mapsto \infty} \frac{(3x+1)^2}{x^2} = \lim_{x \mapsto \infty} \frac{9x^2 + 6x + 1}{x^2} = \lim_{x \mapsto \infty} \frac{9x^2}{x^2} = 9$$

(one could also use Hospital for this... but the rule is even easier: just retain the highest power)

Let us now study the integral

$$\int_{x_0}^{\infty} f(x)\,dx \qquad \text{whereas} \qquad \lim_{x \mapsto \infty} \frac{f(x)}{x^\alpha} = L \neq 0.$$

The question is: is the integral convergent?

For $x$ very large, we can approximate the function as

$$f(x) = L x^\alpha$$

for $x > \Lambda$, where $\Lambda$ is some (properly) chosen large number.

Then:

$$\int_{x_0}^{\infty} f(x)\,dx = \int_{x_0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$\simeq \text{const} + L\underbrace{\int_1^{\infty} x^{\alpha}\,dx}_{\text{is this number finite or infinite?}}$$

$$\int_1^{\infty} x^{\alpha}\,dx = L\left[\frac{x^{\alpha+1}}{\alpha+1}\right]_1^{\infty} \quad \text{for } \alpha \neq -1.$$

Now, for $\alpha+1>0$, $\alpha>-1$ we obviously get $\infty$ as a result, and is finite for $\alpha<-1$. In fact:

$$\lim_{x\to\infty} \frac{x^{\alpha+1}}{\alpha+1} = \begin{cases} =\infty & \text{for } \alpha>-1 \\ 0 & \text{for } \alpha<-1 \end{cases}$$

Ergo:

$$\int_1^{\infty} x^{\alpha}\,dx = \begin{cases} L\cdot\infty & \text{for } \alpha>-1 \\ -L\dfrac{1^{\alpha+1}}{\alpha+1} & \text{for } \alpha<-1 \quad \text{(finite)!} \end{cases}$$

Let us finally study the case $\alpha=1$:

$$\int_1^{\infty} \frac{1}{x}\,dx = L\left[\ln x\right]_1^{\infty} = \lim_{x\to\infty} \ln x - \ln 1 = \infty.$$

Summarizing: the integral diverges for $\boxed{\alpha\geq-1}$ and converges for $\alpha<1$.

Next, we say that the function "behaves as $x^\alpha$" for $x \mapsto 0^+$ if

$$\lim_{x \mapsto 0^+} \frac{f(x)}{x^\alpha} = L \neq 0.$$

For instance, $f(x) = \frac{\sin x}{x^2}$ behaves as $x^{-1}$; In fact:

$$\lim_{x \mapsto 0} \frac{\left(\frac{\sin x}{x^2}\right)}{x^{-1}} = \lim_{x \mapsto 0} \frac{\sin x}{x} = 1$$

Let us study

$$\int_0^{x_0} f(x)\,dx \quad \text{where} \quad \lim_{x \mapsto 0} \frac{f(x)}{x^\alpha} = L \neq 0.$$

The limit means that $f(x) \sim L x^\alpha$ for $x \mapsto 0^+$.

Let us now split the integral as

$$\int_0^{\varepsilon} f(x)\,dx + \underbrace{\int_\varepsilon^{x_0} f(x)\,dx}_{\text{Const}}$$

where $\varepsilon$ is some small number for which $f(x) \simeq L x^\alpha$ holds for $0 < x < \varepsilon$.

$$\int_0^{\varepsilon} f(x)\,dx \simeq \int_0^{\varepsilon} L x^\alpha \underset{\alpha \neq -1}{=} L\left(\frac{x^{\alpha+1}}{\alpha+1}\right)\Bigg|_0^{\varepsilon}$$ ; When taking the limit $x \mapsto 0$

we get:

$$L\left(\frac{\varepsilon^{\alpha+1}}{\alpha+1}\right) - \lim_{x \mapsto 0^+} L\left(\frac{x^{\alpha+1}}{\alpha+1}\right)$$

Ergo, the integral converges for $\alpha + 1 > 0 \Rightarrow \alpha > -1$.

it diverges for $\alpha < -1$.

When studying the case $\alpha = -1$ it is easy to see that it diverges.

summary of all the discussion:

$$f(x) \sim x^{\alpha} \text{ for } x \mapsto +\infty : \int_{x_0}^{\infty} f(x)\,dx \begin{cases} \to \text{converges for } \alpha < -1 \\ \to \text{diverges for } \alpha \geq -1 \end{cases}$$

$$f(x) \sim x^{\alpha} \text{ for } x \mapsto 0 : \int_{0}^{x_0} f(x)\,dx \begin{cases} \to \text{converges for } \alpha > -1 \\ \to \text{diverges for } \alpha \leq -1. \end{cases}$$

<u>Note</u>:
The latter case can be easily repeated when a function has a singularity for $x = x_1$.

$\int_{x_1}^{x_0 > x_1} f(x)\,dx$ converges if $f(x) \sim (x - x_1)^{\alpha}$ and $\alpha > -1$.