Matrices with complex numbers

$$A = \begin{pmatrix} 1 & 3+2i \\ 5+7i & i \end{pmatrix}$$

We can define in the very same way A^t, $\det A$, $\mathrm{Tr}\, A$, eigenvalues, eigenv

but there is a new operation:

$$A^\dagger = (A^t)^*$$

$$A^t = \begin{pmatrix} 1 & 5+7i \\ 3+2i & i \end{pmatrix}$$

$$(A^t)^* = \boxed{\begin{pmatrix} 1 & 5-7i \\ 3-2i & -i \end{pmatrix} = A^\dagger} \Rightarrow \text{obviously: } (A^\dagger)^\dagger = A.$$

A matrix A is hermitian if

$$\boxed{A^\dagger = A}$$

In 2×2 that means:

$$A = \begin{pmatrix} \alpha & \gamma + i\delta \\ \gamma - i\delta & \beta \end{pmatrix}$$ where $\alpha, \beta, \gamma, \delta$ are real numbers.

In fact:

$$A^t = \begin{pmatrix} \alpha & \gamma - i\delta \\ \gamma + i\delta & \beta \end{pmatrix}$$

$$(A^t)^* = A^\dagger = \begin{pmatrix} \alpha & \gamma + i\delta \\ \gamma - i\delta & \beta \end{pmatrix} = A.$$

The eigenvalues of $A = A^\dagger$ are real!!!

$$Av = \lambda v \rightarrow \lambda = v^\dagger A v$$

But then:

$$\lambda^* = \lambda^\dagger = (v^\dagger A v)^\dagger = v^\dagger A^\dagger v = v^\dagger A v = \lambda \Rightarrow \lambda^* = \lambda$$

$$\Downarrow$$

λ real.

Now, we can construct in general for a matrix A the following diag. matrix U:

$$U^\dagger A U = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

whereas, in general, $\lambda_i \in \mathbb{C}$ ... but λ_i are <u>reals</u> if $A^\dagger = A$.

$$U = \begin{pmatrix} v_1 & \cdots & v_N \end{pmatrix}$$

$U^\dagger U = 1_N$.

$M = \{ U \,/\, U^\dagger U = U U^\dagger = 1_N \}$

$SU(N) = (M, \cdot)$ is a group.

U is called a unitary matrix !!!

Take a complex vector $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix}$.

$U\vec{v}$ is just as long as $\vec{v}$ $\Rightarrow$ it is a "generalised" rotation.

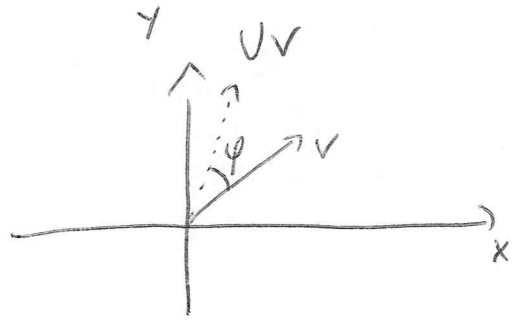
(IMPORTANCE IN QM and QFT).

$N = 1$

$U / U^\dagger U = 1 \Rightarrow U = e^{i\varphi}$

$\vec{v} = v$ is a complex number.

$U\vec{v} = e^{i\varphi} v = w$.



$\|w\| = \|v\|$

it is just a rotation!!!

$U(1) \sim SO(2)$ $\quad e^{i\varphi} \leftrightarrow \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$

$N = 2$

$$U = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$$ with $|\alpha|^2 + |\beta|^2 = 1$.

$$\begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}\begin{pmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} |\alpha|^2+|\beta|^2=1 & \alpha\beta^* - \beta^*\alpha = 0; \\ \beta\alpha^* - \alpha^*\beta = 0; & |\beta|^2+|\alpha|^2 = 1 \end{pmatrix}$$

$$U = e^{iA}$$

$$U^+ = e^{-iA^+} \Rightarrow \text{suppose that } A \text{ is hermitian} : A^+ = A$$

$$U^+ = e^{-iA} \Rightarrow U^+U = UU^+ = 1_2 \;!!!$$

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \underbrace{A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{Pauli-matrices}}$$

$$\boxed{U = e^{i\alpha_i A_i}} \neq e^{i\alpha_0 A_0} e^{i\alpha_1 A_1} e^{i\alpha_2 A_2} e^{i\alpha_3 A_3}$$

exponential form