Differential equations

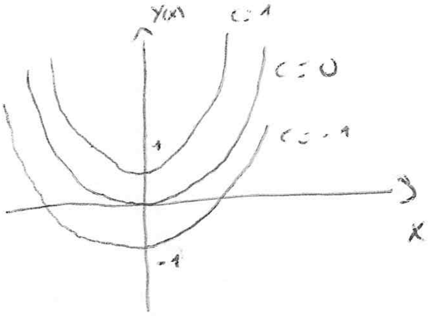
$$y'(x) = 3x$$

is an example of a differential equation.

The class of solutions of this d.e. can be easily obtained by integration:

$$y(x) = \int 3x\,dx = \frac{3x^2}{2} + c$$

where c is a constant.



We thus have an ∞ of solutions.

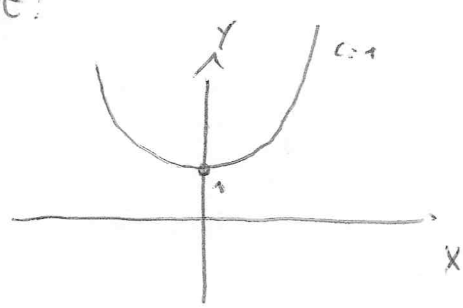
If however we study the system:

$$\begin{cases} y'(x) = 3x \\ y(0) = 1 \end{cases}$$

$\longrightarrow$ THIS IS CALLED INITIAL CONDITION

only one unique solution is left:

$$\boxed{y(x) = \frac{3}{2}x^2 + 1}$$



Of course, in general we may have more difficult diff. eqs. than the studied one :)

In general, a differential equation of the n-th order takes the form

$$F(x, y, y', \dots, y^{(n)}) = 0$$

N.b.: the order of a diff. eq. is given by the highest derivative of the function $y(x)$ present in the equation.

$$\begin{cases} y' = 3 & \rightarrow \text{first order} \\ y^{(5)} + y''' = 1 & \text{5-th order} \end{cases}$$

<u>def</u>: <u>Linear</u> and <u>homog.</u> diff. eq. of the n-th order is given by

$$\boxed{a_0(x)\, y(x) + a_1(x)\, y'(x) + \dots + a_n(x)\, y^{(n)}(x) = 0}$$

if $y_1(x)$ and $y_2(x)$ are solution of this eq $\rightarrow \alpha y_1 + \beta y_2$ is also a solution.

In general, such a diff. eq. admits n indipendent solutions

$y_1(x), y_2(x), \dots, y_n(x)$.

[Recall:

Two function $f(x)$ and $g(x)$ are indipendent if

$$\alpha f(x) + \beta g(x) = 0 \quad \forall x \longmapsto \alpha = \beta = 0$$
]

Note, the general solution of the diff. eq. reads

$$y(x) = \alpha_1 y_1 + \alpha_2 y_2 + \dots + \alpha_n y_n$$

where $\alpha_1, \dots, \alpha_n$ are <u>real</u> constants.

For each choice of $\alpha_1, \dots, \alpha_n$ one has a different solution.

The solution of the diff. eq. is <u>unique</u> if, in addition to the diff. eq., we also have n "initial" conditions, usually put in the form

$$y(x_0) = c_0$$

$$y'(x_0) = c_1$$

$$\dots$$

$$y^{(n-1)}(x_0) = c_{n-1}$$

(Alternatively, boundary conditions like $y(x_1) = c_1, \dots y(x_n) = c_n$ also fix univocally the solution).

Example:

$$y''(x) = 0$$

homog. eq of the 2° order.

$y_1(x) = 1$ is a solution: $y' = 0$, $y'' = 0$ ok.

$y_2(x) = x$ is also a solution: $y' = 1$, $y'' = 0$.

General solution:

$$y(x) = \alpha_1 y_1(x) + \alpha_2 y_2(x) = \alpha_1 + \alpha_2 x$$

is a solution:

$$y'(x) = \alpha_2$$

$$y''(x) = 0.$$

Suppose now that we have the initial condition:

$$\begin{cases} y(0) = 3 \\ y'(0) = 2 \end{cases}$$

$$y(x) = \alpha_1 + \alpha_2 x \rightarrow \begin{array}{l} y(0) = \alpha_1 = 3 \\ y'(x) = \alpha_2 = 2 \end{array}$$

The unique solution is then:

$$\boxed{y(x) = 3 + 2x}$$

Alternatively, we could also have had the following initial conditions:

$$\begin{cases} y(0) = 2 \\ y(1) = 4 \end{cases}$$

$$y(x) = \alpha_1 + \alpha_2 x$$

$$y(0) = \alpha_1 = 2$$

$$y(1) = \alpha_1 + 4\alpha_2 = 4 \rightarrow 2 + 4\alpha_2 = 4 \rightarrow 4\alpha_2 = 2$$

$$\boxed{\alpha_2 = 1/2}$$

Ergo, in this case the unique solution is:

$$y(x) = 2 + \frac{x}{2}$$

Check:

$$y(0) = 2$$

$$y(1) = 2 + \frac{4}{2} = 2 + 2 = 4 \quad ✓$$

An <u>inhomogeneous</u> linear diff. eq. takes the form

$$a_0(x)\,y(x) + a_1(x)\,y'(x) + \dots + a_n(x)\,y^{(n)}(x) = \underset{\sim}{q(x)} \neq 0$$

($q(x)=0$ was the hom. case)

Consider now:

$$y_{hom}(x) = \alpha_1 y_1(x) + \dots + \alpha_n y_n(x)$$

is the solution of the homog. eq. (for each $\alpha_1, \dots \alpha_n \in \mathbb{R}$).

consider then a "special" solution of the form $y_{spec}(x)$, which fulfills

$$a_0(x)\,y_{spec}(x) + \dots + a_n(x)\,y^{(n)}_{spec}(x) = q(x).$$

Then, the function:

$$y_{full} = y_{hom}(x) + y_{spec}(x)$$

is a solution of the full diff. eq.

[To see it: Just plug in...]

$$y_{full}(x) = \alpha_1 y_1 + \dots + \alpha_n \quad + y_{spec}(x)$$

still depends on "n" real constants, which are univocally fixed by n initial condition, just as in the homog. case.

Consider the inhom. equation

$$y''(x) = x \quad (q(x) \neq 0)$$

$$y_{hom}(x) = \alpha_1 + \alpha_2 x.$$

$$y_{spec}(x) = a x^3 \qquad y'_{spec}(x) = 3 a x^2 \qquad y''_{spec}(x) = 6 a x = x$$

$$\Downarrow$$

$$a = 1/6$$

Ergo, the most general solution is:

$$\boxed{y(x) = \alpha_1 + \alpha_2 x + \frac{x^3}{6}}$$

Suppose that, in addition, we have the initial condition

$$y(0) = 3$$

$$y'(0) = 2$$

$$\Rightarrow y(x) = 3 + 2x + \frac{x^3}{6}$$

if on the other hand we start from

$$\begin{cases} y(0) = 2 = \alpha_1 & \to \alpha_1 = 2 \\ y(1) = 4 = \alpha_1 + \alpha_2 + \frac{1}{6} & \to 2 + \alpha_2 + \frac{1}{6} = 4 \to \alpha_2 = 2 - \frac{1}{6}; \end{cases}$$

$$\boxed{\alpha_2 = \frac{11}{6}}$$

$$\boxed{y(x) = 2 + \frac{11}{6} x + \frac{x^3}{6}}$$

Non-linear eq. are in general much more complicated, see for instance

$y' + y^3 = 0$

or

$y' - 2^y = 0$

...

We will not consider in the following such cases...

Note that non-linear diff. eq. are much more complicated.

In fact, having two solutions, you cannot construct further solutions by summing them.

Linear diff. eq. of the first order

$$\boxed{y'(x) + p(x)\,y = q(x)}$$

A unique solution is obtained if, in addition, you have the (one) initial condition

$$y(x_0) = y_0 .$$

Trivial case

$$p(x) = q(x) = 0 .$$

$$y'(x) = 0 .$$

solution: $y_1(x) = 1$ is such.

General solution:

$$y(x) = \alpha\, y_1(x) = y_0 .$$

Together with the initial condition $y(x_0) = y_0$, we get the unique solution

$$\boxed{y(x) = y_0 .}$$

Simple case : $p(x) = 0$.

$$y'(x) = q(x).$$

By integrating we get

$$y(x) = \int q(x)\,dx = Q(x) + c$$

where $Q(x)$ is a primitive function of $q(x)$.

In this case, the solution of the diff. eq. amounts to solve an integral.

If we have also $y(x_0) = y_0$ we can write the unique solution as

$$\boxed{y(x) = \int_{x_0}^{x} q(z)\,dz + y_0}$$

in fact $y'(x) = q(x)$ and $y(x_0) = y_0$ are fulfilled.

In general, $y_{hom}(x) = 1$ is a solution of the hom. equation.

$y_{spec}(x) = \int_{x_0}^{x} q(z)\,dz$ represents a special solution of the full eq.

Therefore: $y(x) = y_{spec}(x) + \alpha\, y_{hom}(x)$ as in the general discussion before

Homogen. case : $q(x) = 0$.

$$y'(x) + p(x)\, y(x) = 0.$$

$$\frac{dy(x)}{dx} = -p(x)\, y(x)$$

$$\frac{dy(x)}{y(x)} = -p(x)\, dx$$

$$\int \frac{dy}{y} = -\int p(x)\, dx$$

$$\ln y = -\int p(x)\, dx$$

$$\boxed{y = e^{-\int p(x)\, dx}}$$

$$\boxed{P(x) = \int_{x_0}^{x} p(x)\, dx}$$

$$\int p(x) dx = P(x) + c$$ with $P(x) = p(x)$

Therefore

$$y(x) = e^{-c} \cdot e^{-P(x)}$$ where α is a constant.

Verify that it works:

$$y(x) = e^{-c} \cdot e^{-P(x)} \quad \text{with } P'(x) = p(x)$$

Let us check that it is a solution:

$$y'(x) = -e^{-c} \cdot P'(x) e^{-P(x)} = p(x) e^{-c} \cdot e^{-P(x)}$$

$y'(x) + p(x)\, y(x) =$

$$= -e^{-c} p(x) e^{-P(x)} + p(x) e^{-c} e^{-P(x)} = 0 \quad \checkmark$$

OK

With initial condition:

$$y(x_0) = y_0$$

$$y_0 = e^{-c} \, e^{P(x_0)}$$

$$\ln y_0 = -c + P(x_0) \mapsto \quad c = P(x_0) - \ln y_0 .$$

In this case we can write from the very beginning the unique solution by writing

$$\frac{dy}{y} = -p(x)\,dx$$

$$\int_{y_0}^{y} \frac{dy}{y} = -\int_{x_0}^{x} p(x)\,dx$$

$$\ln\left(\frac{y}{y_0}\right) = -\int_{x_0}^{x} p(x)\,dx$$

$$\boxed{y = y_0 \, e^{-\int_{x_0}^{x} p(x)\,dx}}$$

$y(x_0) = y_0$ fulfilled.

Example:

$p(x) = \lambda \quad \rightarrow \quad P(x) = \lambda x$

$$\begin{cases} y'(x) + y(x) = 0 \\ y(x_0) = y_0 \end{cases}$$

Ergo:

$$y(x) = y_0 \, e^{-\int_{x_0}^{x} \lambda \, dx} = y_0 \, e^{-\lambda (x - x_0)}$$

$$\boxed{y(x) = y_0 \, e^{-\lambda (x - x_0)}}$$

Physical Example

$y \mapsto N \quad (y_0 \mapsto N_0)$

$x \mapsto t$

$x_0 \mapsto t_0 = 0$

$\frac{dN}{dt} = -\lambda N \mapsto$ Decay law as function of time:

$$N(t) = N_0 \, e^{-\lambda t}$$

$\tau = \frac{1}{\lambda}$ is called "mean life" (which, by the way, is not the half-life)

Full case : $p(x) \neq 0$, $q(x) \neq 0$

$$y'(x) + p(x)y(x) = q(x)$$

In order to solve it multiply by $e^{-\int p(x)dx}$ from the left and from the right:

$$\underbrace{y'(x)\, e^{-\int p(x)dx} + p(x)y(x)\, e^{-\int p(x)dx}}_{} = q(x)\, e^{\int p(x)dx}$$

$$\frac{d}{dx}\left(y(x)\, e^{\int p(x)\,dx} \right) = q(x)\, e^{\int p(x)dx}$$

$$y(x)\, e^{\int p(x)dx} = \int q(x)\, e^{\int p(x)dx}\, dx$$

Ergo:

$$y(x) = \left[\int q(x)\, e^{\int p(x)dx}\, dx \right] e^{-\int p(x)dx}$$

One should be careful with the "initial conditions"...

Consider:

$P(x)\ /\ P'(x) = p(x)$

$$W(x)\ /\ W'(x) = q(x)\, e^{P(x)}$$

The most general solution can be written as:

$$y(x) = [\underbrace{W(x) + c}]\, e^{-P(x)}$$

unique constant...

$$y(x_0) = y_0 = c\, e^{-P(x_0)} \rightarrow \boxed{c = y_0\, e^{P(x_0)}}$$

Note:

$$y(x) = \underbrace{W(x)\, e^{-P(x)}}_{y_{spec}(x)} + \underbrace{c\, e^{-P(x)}}_{y_{hom}(x)}$$

as in the general case studied before.

Nb: very often one writes the most general solution in the following form:

$$y(x) = \left[\int q(x)\, e^{\int p(x)\,dx}\, dx + c \right] e^{-\int p(x)\,dx}$$

This is mathematically not fully consistent, because the integrals are indefinite and therefore include the additional constant...

However, it is good for mnemonic.

Example:

$$y'(x) + \lambda y(x) = \beta$$

$[p(x) = \lambda = \text{const}, \quad q(x) = \beta = \text{const}]$

$$\int p(x)dx = \int \lambda \, dx = \lambda x + \text{const} \to P(x) = \lambda x.$$

$$\int q(x) e^{\int p(x)dx} dx = \int \beta e^{\lambda x} dx = \frac{\beta}{\lambda} e^{\lambda x} + \text{const} \to W = \frac{\beta}{\lambda} e^{\lambda x}$$

$$y(x) = [W + c] e^{-P(x)} = \left[\frac{\beta}{\lambda} e^{\lambda x} + c\right] e^{-\lambda x} = \frac{\beta}{\lambda} + c e^{-\lambda x}$$

[check:

$$y' = -\lambda c e^{-\lambda x}$$

$$-c\lambda e^{-\lambda x} + \lambda \left(\frac{\beta}{\lambda} + \lambda c e^{-\lambda x}\right) = \beta \quad \checkmark \text{ok}$$

]

$$\boxed{y(x) = \frac{\beta}{\lambda} + c e^{-\lambda x}}$$

if $y(x_0) = y_0 = \frac{\beta}{\lambda} + c e^{-\lambda x_0} \to \left(y_0 - \frac{\beta}{\lambda}\right) = c e^{-\lambda x_0}$

$$\Downarrow$$

$$\boxed{c = e^{\lambda x_0}\left(y_0 - \frac{\beta}{\lambda}\right)}$$

Basic, always present, diff. eq. in physics:

$$\boxed{y'' + \omega^2 y = 0} \qquad (\omega \in \mathbb{R}^+) \qquad \text{Harmonic-Oscill. eq.}$$

TRY WITH

$y(x) = \sin(ax)$

$y'(x) = a\cos(ax)$

$y''(x) = -a^2 \sin(ax)$

$\Downarrow$

$-a^2 \sin(ax) + \omega^2 \sin(ax) = 0 \quad \rightarrow \quad a^2 = \omega^2 \rightarrow \boxed{a = \omega}$

($a = -\omega \rightarrow$ just a minus sign).

$y_1(x) = \sin(\omega x)$ is a solution.

Similarly:

$y_2(x) = \cos(\omega x)$ is also a solution.

$y_1(x)$ and $y_2(x)$ are <u>linearly indipendent</u>.

The most general solution is:

$$y(x) = \alpha_1 y_1(x) + \alpha_2 y_2(x) = \alpha_1 \sin(\omega x) + \alpha_2 \cos(\omega x).$$

$Y(0) = Y_0$

$Y'(0) = 0$

$$\begin{cases} Y(x) = \alpha_1 \sin(\omega x) + \alpha_2 \cos(\omega x) \\ Y(0) = \alpha_2 = Y_0 \end{cases}$$

$$\begin{cases} Y'(x) = \omega \alpha_1 \cos(\omega x) - \alpha_2 \omega \sin(\omega x) \\ Y'(0) = \omega \alpha_1 = 0 \longmapsto \alpha_1 = 0 \end{cases}$$

$$Y(x) = Y_0 \cos(\omega x)$$

is the unique solution in this case.

Physical meaning:

this case corresponds to the "very famous" harmonic oscillator:

$x \longmapsto t$

$Y \longmapsto X$ $\quad \Rightarrow \quad \ddot{X}(t) + \omega^2 X = 0$

$Y' \longmapsto \dot{X}$

The particles moves back and forward with 'frequency' ω.

This equation appears everywhere... indeed, if you can reduce a difficult problem to an harmonic oscillator you can say that you solved it.

Let us add a constant inhom. term:

$$y'' + \omega^2 y = K = \text{const.}$$

A special solution is easily found as:

$$y_{spec}(x) = \frac{K}{\omega^2} = \text{const.} \qquad (y'_{spec} = y''_{spec} = 0).$$

$$\boxed{y(x) = \alpha_1 \sin(\omega x) + \alpha_2 \cos(\omega x) + \frac{K}{\omega^2}}$$

is the most general solution.

Let us now invert the sign and study the following diff. eq:

$$\boxed{y'' - \beta^2 y = 0}$$

obviously, $\sin(\beta x)$ is <u>not</u> a solution.

In fact, you get:

$y = \sin(\beta x) \rightarrow y' = \beta\cos(\beta x) \quad y''(x) = -\beta^2 \sin\beta x$

$-\beta^2 \sin\beta x - \beta^2 \sin\beta x \neq 0$

[One could get the correct solution by considering β being imaginary...]

Let us search for different solutions:

$y(x) = e^{ax}$

$y' = a e^{ax}$

$y'' = a^2 e^{ax}$

$\rightarrow y'' - \beta^2 y \rightarrow a^2 e^{ax} - \beta^2 e^{ax} = 0 \rightarrow a = \pm\beta$

$$\boxed{y_1(x) = e^{\beta x} , \quad y_2(x) = e^{-\beta x}}$$

$y = \alpha_1 y_1 + \alpha_2 y_2 = \alpha_1 e^{\beta x} + \alpha_2 e^{-\beta x}$ is the most general solution.

it is customary to express this solution by using the following functions:

$$\begin{cases} \cosh x = \frac{1}{2}(e^{x} + e^{-x}) \\ \sinh x = \frac{1}{2}(e^{x} - e^{-x}) \end{cases}$$

$$\begin{cases} y_1(x) = \cosh(\beta x) \\ y_2(x) = \sinh(\beta x) \end{cases}$$ are two indep. solutions.

$$y = \alpha_1 \cosh(\beta x) + \alpha_2 \sinh(\beta x)$$

represents another possible way to express the most general solution.

$\cosh x$ and $\sinh x$ are the "divergent analogous" of the function $\sin x$ and $\cos x$. A close relationship is obtained by using complex numbers.

Physical example: chain on the table falling of because of gravity.

It falls faster and faster