Eigenvalues and Eigenvectors

Let us consider the $N \times N$ real matrix A.

We then study the equation

$$A \cdot \vec{v} = \lambda \vec{v}$$

where $\vec{v} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$ is a vector and λ is a real number (called eigenvalue).

¡ COLUMN!

Of course

$v = \vec{0} \left(\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \right)$ is a solution of that eq. for each λ.

Indeed, we search $\boxed{\vec{v} \neq \vec{0}}$ such that

$$A \vec{v} = \lambda \vec{v}$$

$\vec{v}$ that fullfills this eq. is called "eigenvector" and λ is the corresponding eigenvalue.

We can rewrite the equation as

$$\underbrace{(A - \lambda \cdot 1_N)}_{\text{This is a } N\times N \text{ matrix } N} \vec{v} = \vec{0}$$

Example:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$A - \lambda \cdot 1_2 = \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix}$$

Now, let us write $M = A - \lambda \cdot 1_N$, then the eigenvalue eq. reads:

$$M \vec{v} = \vec{0}.$$

If M <u>has</u> an inverse, M^{-1}, we can multiply by M^{-1} from the left:

$$M^{-1} M \vec{v} = \vec{v} = M^{-1} \vec{0} \Rightarrow \vec{v} = \vec{0}$$

"trivial solution", which we are "not" interested!

But M^{-1} exists if $\det M \neq 0$.

Then, in this case $\left(\det M = \det(A - \lambda 1_N) \neq 0 \right)$

the only solution is $\vec{v} = \vec{0}$ ...

in which we are "<u>not</u>" interested in.

On the contrary, the interesting case is realized for

$$\det M = \boxed{\det(A - \lambda \cdot 1_N) = 0}$$

For values of λ such that the previous eq. is fullfilled, an Eigenvector $\vec{v} \neq \vec{0}$ such that $A\vec{v} = \lambda \vec{v}$ can be found.

Note, if $\vec{v}$ fullfills the eq. $A\vec{v} = \lambda\vec{v}$, then also $c\vec{v}$ is such.

In general, we can therefore take $\vec{v}$ with $\|\vec{v}\| = 1$.

$\det(A - \lambda \cdot 1_N)$ is an algebraic eq. of order N.

There is how a problem: not always one has N real solutions.

$\lambda^2 + 1 = 0 \rightarrow$ No real solution...

However, if A is a symmetric matter, we are sure that only real eigenvalues exist.

Let us show it for $N=2$:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

$$A - \lambda \cdot 1_2 = \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{12} & a_{22} - \lambda \end{pmatrix}$$

The eigenvalue eq. reads

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}^2 = 0$$

$$\lambda^2 - \lambda(a_{11} + a_{22}) + a_{11}a_{22} - a_{12}^2 = 0$$

$$\lambda_{1,2} = \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}^2)}}{2}$$

$$= \frac{a_{11} + a_{22} \pm \sqrt{a_{11}^2 + a_{22}^2 + 2a_{11}a_{22} - 4a_{11}a_{22} + 4a_{12}^2}}{2}$$

$$= \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}^2}}{2}$$

$\Delta > 0 \longrightarrow \lambda_1$ and λ_2 are surely real numbers.

If A is a symmetric matrix, we have N real eigenvalues

$\lambda_1, \dots \lambda_N$

and N real eigenvectors $\vec{v}_i \neq \vec{0}$. We can chose them

having length 1 and being orthogonal to each other:

$$\begin{cases} |\vec{v}_i| = 1 \; ; \; i = 1, \dots, N \\ \vec{v}_i \cdot \vec{v}_j = 0 \qquad i \neq j \end{cases}$$

$$\Downarrow$$

$$\boxed{\vec{v}_i \cdot \vec{v}_j = \delta_{ij}}$$

Note that the scalar product of two vectors can be easily written as a "matrix" product:

$$\vec{V}_1 \cdot \vec{V}_2 = \begin{pmatrix} V_1^{(1)} \\ \vdots \\ V_1^{(N)} \end{pmatrix} \cdot_{SP} \begin{pmatrix} V_2^{(1)} \\ \vdots \\ V_2^{(N)} \end{pmatrix} =$$

$$= \underbrace{\vec{V}_1^{\,t}\,\vec{V}_2}_{\substack{\text{usual matrix-multiplication} \\ \text{(No need of any simble ...)}}} = (V_1^{(1)}, \dots, V_1^{(N)}) \cdot \begin{pmatrix} V_2^{(1)} \\ \vdots \\ V_2^{(N)} \end{pmatrix} = V_1^{(1)} V_2^{(1)} + \dots V_1^{(N)} V_2^{(N)}$$

Let us show now that, if we consider two eigenvalues λ_1 and λ_2 with $\lambda_1 \neq \lambda_2$, then the corresponding eigenvectors $\vec{V}_1$ and $\vec{V}_2$ are orthogonal:

$$\boxed{\vec{V}_1 \cdot \vec{V}_2 = 0}$$

$$\vec{V}_1^{\,t}\,\vec{V}_2 = 0$$

$$\begin{cases} A\vec{v}_1 = \lambda_1 \vec{v}_1 \\ A\vec{v}_2 = \lambda_2 \vec{v}_2 \end{cases} \Rightarrow (A\vec{v}_2)^t = \vec{v}_2^{\,t} A^t = \vec{v}_2^{\,t} A \quad \text{because } A = A^t$$

Let us then study:

$$\vec{v}_2^{\,t} A \vec{v}_1 = \vec{v}_2^{\,t} \underbrace{(A\vec{v}_1)}_{\lambda_1 \vec{v}_1} = \lambda_1 \vec{v}_2^{\,t} \vec{v}_1 = \lambda_1 (\vec{v}_2 \cdot \vec{v}_1) = \lambda_1 (\vec{v}_1 \cdot \vec{v}_2)$$

$$= \underbrace{(\vec{v}_2^{\,t} A)}_{\lambda_2 \vec{v}_2^{\,t}} \vec{v}_1 = \lambda_2 \vec{v}_2^{\,t} \vec{v}_1 = \lambda_2 (\vec{v}_1 \cdot \vec{v}_2)$$

Ergo:

$$\lambda_1 (\vec{v}_1 \cdot \vec{v}_2) = \lambda_2 (\vec{v}_1 \cdot \vec{v}_2)$$

$$\Downarrow$$

$$(\lambda_1 - \lambda_2)(\vec{v}_1 \cdot \vec{v}_2) = 0$$

$$\lambda_1 \neq \lambda_2 \Rightarrow \boxed{\vec{v}_1 \cdot \vec{v}_2 = 0}$$

Now, let us do

Orthogonal matrices

Let us start from $A\vec{v}_i = \lambda_i \vec{v}_i$, $i = 1, \dots N$ and $\vec{v}_i \cdot \vec{v}_j = \vec{v}_i^{\,t} \vec{v}_j = \delta_{ij}$.

$A^t = A \to \lambda_i$ and $\vec{v}_i$ are reals...

We construct the matrix B as it follows:

$$B = \begin{pmatrix} \vec{v}_1 & \cdots & \vec{v}_N \end{pmatrix} = \begin{pmatrix} v_1^{(1)} & v_2^{(1)} & & v_N^{(1)} \\ \vdots & & & \\ v_1^{(N)} & v_2^{(N)} & & v_N^{(N)} \end{pmatrix}$$

Then, it is "tedious" but straightforward to prove that:

$$\boxed{BB^t = B^t B = 1_N = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix}}$$

In fact

$$B^t B = \begin{pmatrix} \vec{v}_1^{\,2} = 1 & \overbrace{\vec{v}_1 \cdot \vec{v}_2}^{0} & & \\ \underbrace{\vec{v}_2 \cdot \vec{v}_1}_{0} & \vec{v}_2^{\,2} = 1 & & \\ & & \ddots & \\ & & & \vec{v}_N^{\,2} = 1 \end{pmatrix}$$

B fulfilling $B^t B = B B^t = 1$ is called "orthogonal"

From the definitions it also follows that

$$A \cdot B = B D \qquad \text{where } D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_N \end{pmatrix}$$

Mutiplying with B^t from the left we find:

$$B^t A B = D.$$

One says that B diagonalises A.

Moreover:

$$\det(B^t A B) = \det(B)^2 \det A = \det A = \det D = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_N$$

$$\mathrm{Tr}[B^t A B] = \mathrm{Tr}[B B^t A] = \mathrm{Tr}[A] = \mathrm{Tr}[D] = \lambda_1 + \lambda_2 + \ldots + \lambda_N$$

Consider now a vector $\vec{v}$.

The length of $\vec{v}$ is given by $\|\vec{v}\| = \sqrt{v_1^2 + \dots v_N^2} = \sqrt{\vec{v}^t \vec{v}}$

Now, let B be an orthogonal matrix and consider:

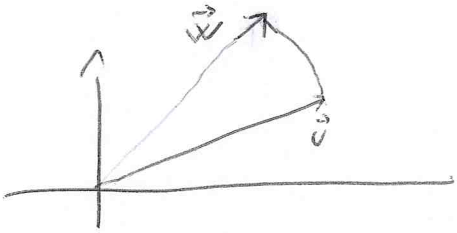
$$\vec{w} = B\vec{v}$$

How long is $\vec{w}$?

$$\vec{w}^t = \vec{v}^t B^t$$

$$\|\vec{w}\|^2 = \sqrt{\vec{w}^t \vec{w}} = \sqrt{\vec{v}^t B^t B \vec{v}} = \sqrt{\vec{v}^t \vec{v}} = \|\vec{v}\|$$

$\vec{w}$ has the same length of $\vec{v}$ ... we are doing a "rotation":



This is a very important property (VIP): the orthogonal matrix B leaves the length invariant!!!

Group $O(N)$

$$M = \{ B \text{ is a } N\times N \text{ matrix} \,/\, B^tB = BB^t = 1_N \}$$

$O(N) = (M, \cdot)$ is a group

↳ Matrix mult.

Internal operation:

$B_1 B_2$ is also orthogonal:

$$(B_1B_2)^t(B_1B_2) = B_2^t \underbrace{B_1^t B_1}_{1_N} B_2 = \underbrace{B_2^t B_2}_{1_N} = 1_N .$$

- $B_1(B_2B_3) = B_1(B_2B_3)$
- $1_N \in M$: $1_N^t 1_N = 1_N 1_N^t = 1$.
- $B^tB = BB^t = 1 \Rightarrow B^t = B^{-1}$!!!

B^t is orthogonal:

$$(B^t)^t \cdot B^t = (BB^t)^t = 1 \text{ !!}$$

$$BB^t = 1$$

$$\det(BB^t) = \det B \det B^t = \det B \det B = (\det B)^2 = \det 1_N = 1$$

$$\Downarrow$$

$$\det B = \pm 1$$

Let us consider now only the matrices B / $\det B = +1$

$$M = \{ B \,/\, B^t B = B B^t = 1 \text{ and } \det B = +1 \}$$

$(M, \cdot)$ is also a group $\Rightarrow$ ex.

It is called "$SO(N)$".

SO(2): explicit expression

$$B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad B^t = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$B^t B = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}\begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} c^2+s^2=1 & cs-sc=0 \\ sc-cs=0 & s^2+c^2=1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det B = \cos^2\theta - (-\sin^2\theta) = c^2+s^2 = 1.$$

if we consider, on the contrary:

$$\tilde{B} = \begin{pmatrix} -\cos\theta & \sin\theta \\ \sin\theta & +\cos\theta \end{pmatrix}$$

$$\tilde{B}^t\tilde{B} = \begin{pmatrix} -c & s \\ s & +c \end{pmatrix}\begin{pmatrix} -c & s \\ s & c \end{pmatrix} = \begin{pmatrix} +c^2+s^2=1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\det\tilde{B} = -1$$

$\tilde{B}$ belongs to $O(2)$ but <u>not</u> to $SO(2)$

Function $f(x,y)$ / $\vec{\nabla} f = 0$ for $x = y = 0$. To 2^{o}

$$f(x,y) = f(0,0) + \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)_{\vec{0}} x^2 + \frac{1}{2}\left(\frac{\partial^2 f}{\partial y^2}\right)_{\vec{0}} y^2 + \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\vec{0}} xy$$

$$= f(0,0) + \frac{1}{2}(x, y) \begin{pmatrix} (\partial_x^2 f)_{\vec{0}} & \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\vec{0}} \\ \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\vec{0}} & (\partial_y^2 f)_{\vec{0}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Now, let us write it as

$$f(x,y) = f(0,0) + \frac{1}{2} \vec{x}^t A \vec{x}.$$

Let us "diagonalise" A with $B = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$:

$$B^t A B = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \quad (B^t B = 1)$$

Then, let us perform the transformation

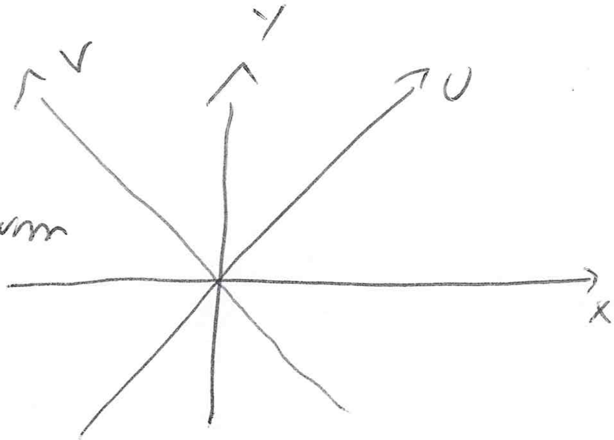
$$\begin{cases} \vec{x} = B\vec{u} \quad \text{with } \vec{u} = \begin{pmatrix} u \\ v \end{pmatrix} \\ \vec{x}^t = \vec{u}^t B^t \end{cases}$$

$$f(x,y) = f(0,0) + \frac{1}{2}\vec{u}^{\,t} B^t A B \vec{u} =$$

$$= f(0,0) + \frac{1}{2}(u,v)\begin{pmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}u \\ v\end{pmatrix} =$$

$$= f(0,0) + \frac{1}{2}\lambda_1 u^2 + \frac{1}{2}\lambda_2 v^2$$ $\Rightarrow$ No "uv term here"...

$$\begin{cases} \lambda_1, \lambda_2 > 0 \Rightarrow u=v=0 \text{ is a minimum} \\ \lambda_1 < 0, \lambda_2 > 0 \Rightarrow \text{saddle} \\ \lambda_1 < 0, \lambda_2 < 0 \Rightarrow \text{Maximum} \end{cases}$$



$\rightarrow$ exercise: do it in a slightly different way...

B orthogonal in exp. form

$B \,/\, B^t B = 1$.

$B = e^A$

which properties should A fullfill?

$B^t = e^{A^t}$

$B^t \cdot B = e^{A^t} \cdot e^A$

Now, suppose that $A^t = -A$ (i.e. A is antisymmetric).

Then:

$B^t B = e^{-A} \cdot e^{A} = e^{-A+A} = e^{0_N} = 1_N$.

$B = e^A$ with $A^t = -A$.

Note that it follows that:

$A^t + A = 0$

$Tr[A^t + A] = 2\, Tr A = 0 \rightarrow Tr A = 0$

Ergo:

$$\det B = e^{\mathrm{Tr} A} = 1_N .$$

$B = e^{A}$ belongs to $SO(N)$ $(\det B = +1)$.

Examples:

$SO(2)$ $\quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$B = e^{\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} \equiv \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$SO(3)$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ +1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$(A_i)_{JK} = \varepsilon_{iJK}$$

$B \in SO(3)$:

$$B = e^{\vartheta_i A_i} = e^{\vartheta_1 A_1 + \vartheta_2 A_2 + \vartheta_3 A_3}$$

$\Rightarrow$ a 3×3 matrix.

$$B = e^{\vartheta_3 A_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta_3 & \sin\vartheta_3 \\ 0 & -\sin\vartheta_3 & \cos\vartheta_3 \end{pmatrix}$$