FOURIER

Let us consider a function $f(x): [-\pi\lambda, \pi\lambda] \longmapsto R$ such that:

$f(-\pi\lambda) = f(\pi\lambda)$ (periodicity)

$f'(-\pi\lambda) = f'(\pi\lambda)$

...

Now, consider $e^{imx/\lambda}$. $m = 0, \pm 1, \pm 2, \dots$

$$\begin{cases} x = -\pi\lambda \\ e^{im(-\pi\lambda)/\lambda} = e^{-im\pi} \end{cases}$$

$$\begin{cases} x = +\pi\lambda \\ e^{im(\pi\lambda)/\lambda} = e^{im\pi} \end{cases}$$

This function $e^{imx/\lambda}$ is periodic. Also its derivative is obviously such.

The system of functions $\{e^{imx/\lambda}, m = 0, \pm 1 \dots\}$ forms an orthogonal and complete set.

$$\int_{-\pi\lambda}^{\pi\lambda} e^{-imx/\lambda} e^{inx/\lambda} dx = \int_{-\pi\lambda}^{\pi\lambda} e^{i(n-m)x/\lambda} dx \overset{n \neq m}{=} \frac{\lambda}{i(n-m)} \left(e^{i(n-m)x/\lambda} \right)_{-\pi\lambda}^{\pi\lambda} = 0$$

and for $m = n$ we get $\int_{-\pi\lambda}^{\pi\lambda} dx = 2\pi\lambda$.

Ergo:

$$\boxed{\int_{-\pi\lambda}^{\pi\lambda} e^{-imx/\lambda} e^{inx/\lambda} dx = 2\pi\lambda \cdot \delta_{mn}}$$

Now, let us write the periodic function $f(x)$ as a series:

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{A_n}{\sqrt{2\pi}} e^{inx/\lambda}$$

where A_n are complex nr.

Obviously, if $f(x)$ is real it must be:

$$f^*(x) = \sum_{n=-\infty}^{\infty} \frac{A_n^*}{\sqrt{2\pi}} e^{-inx/\lambda} = \sum_{n=-\infty}^{\infty} \frac{A_n}{\sqrt{2\pi}} e^{inx/\lambda}$$

Let us multiply from the left with $e^{-imx/\lambda}$ and integrate:

$$\int_{-\pi\lambda}^{\pi\lambda} e^{-imx/\lambda} f(x)\, dx = \sum_{n=-\infty}^{\infty} \frac{A_n}{\sqrt{2\pi}} \underbrace{\int_{-\pi\lambda}^{\pi\lambda} e^{-imx/\lambda} e^{inx/\lambda}\, dx}_{\delta_{mn} 2\pi\lambda}$$

$$= \sum_{n} \frac{A_n}{\sqrt{2\pi}} \delta_{mn} 2\pi\lambda = A_m \sqrt{2\pi}\, \lambda$$

Ergo:

$$A_m = \frac{1}{\sqrt{2\pi}} \frac{1}{\lambda} \int_{-\pi\lambda}^{\pi\lambda} e^{-imx/\lambda} f(x)\, dx$$

Important consequence:

$$\frac{1}{\sqrt{2\pi}\,\lambda}\int_{-\pi\lambda}^{\pi\lambda}|f(x)|^2\,dx = \frac{1}{\sqrt{2\pi}\,\lambda}\int_{-\pi\lambda}^{\pi\lambda} f^*(x)\,f(x)\,dx =$$

$$= \frac{1}{\sqrt{2\pi}\,\lambda}\sum_{n,m}\frac{A_m^*}{\sqrt{2\pi}}\frac{A_n}{\sqrt{2\pi}}\underbrace{\int_{-\pi\lambda}^{\pi\lambda} e^{-imx/\lambda}\,e^{inx/\lambda}\,dx}_{2\pi\lambda\,\delta_{mn}}$$

$$= \frac{1}{\sqrt{2\pi}\,\lambda}\sum_n\frac{|A_n|^2}{2\pi}\,2\pi\lambda = \frac{1}{2\pi}\sum_n|A_n|^2$$

$$\frac{1}{2\pi\lambda}\int_{-\pi\lambda}^{\pi\lambda}|f(x)|^2\,dx = \frac{1}{2\pi}\sum_n|A_n|^2\,.$$

$$\boxed{A_n = \frac{1}{\sqrt{2\pi}} \frac{1}{\lambda} \int_{-\pi\lambda}^{\pi\lambda} e^{-inx/\lambda} f(x)\, dx}$$

Let us now consider $\lambda \mapsto \infty$

$$K = \frac{n}{\lambda}$$

$$\lim_{\lambda \mapsto \infty} \lambda A_n = g(K)$$

Ergo we get:

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{A_n}{\sqrt{2\pi}} e^{inx/\lambda} \cdot \left(\frac{\lambda}{\lambda}\right) \qquad dK = \frac{1}{\lambda}$$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(K) e^{iKx}\, dK}$$

$$\boxed{g(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iKx}\, dx}$$

$f(x) : R \mapsto R$

We define its fourier transform as

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

from which it follows that:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(k) e^{ikx} dk$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ikx} \int_{-\infty}^{\infty} f(x') e^{-ikx'} dx'$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x')\, dx' \int_{-\infty}^{\infty} dk\, e^{ik(x-x')}$$

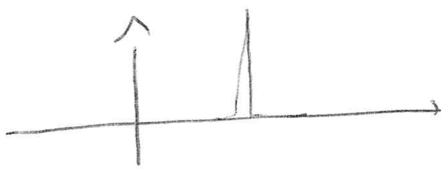
This makes sense only if:

$$\boxed{\frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{ik(x-x')} = \delta(x-x')}$$

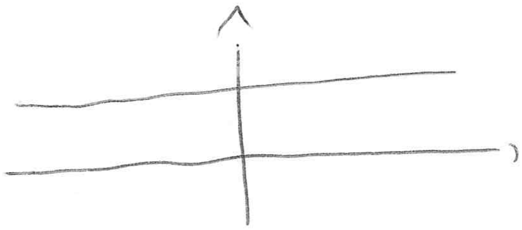
Parseval:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |g(k)|^2 dk$$

<u>Very Important Examples</u>



$$\begin{cases} f(x) = f_0 \delta(x - x_0) \\ g(k) = \dfrac{f_0}{\sqrt{2\pi}} e^{-ikx_0} \end{cases}$$



$$\begin{cases} f(x) = f_0 \\ g(k) = \dfrac{f_0}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-ikx} dx}_{2\pi\,\delta(k)} = \dfrac{f_0}{\sqrt{2\pi}} 2\pi\, \delta(k) = \sqrt{2\pi}\, f_0\, \delta(k) \end{cases}$$

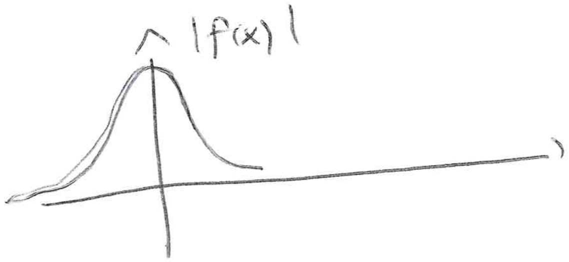
More in general:

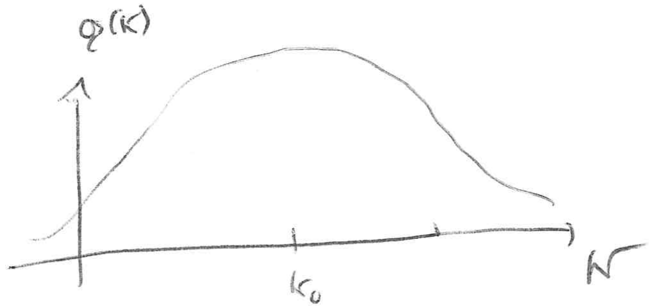
$$\begin{cases} f(x) = f_0 e^{ik_0 x} \\ g(k) = \dfrac{f_0}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} e^{-ikx} e^{ik_0 x} dx = \sqrt{2\pi}\, f_0\, \delta(k - k_0). \end{cases}$$

$$\begin{cases} f(x) = \frac{N}{\sqrt{2\pi}}\, e^{i k_0 x}\, e^{-x^2 \sigma^2/2} \\ g(k) = C\, e^{-(k-k_0)^2/2\sigma^2} \end{cases}$$

The more σ is small ... the more $f(x)$ is peaked

and $g(k)$ "large" ...





"Uncertainty relation in QM"