Group $G = (M, \cdot)$

M = set , $\cdot$ internal operation of the set : $\cdot : G \times G \longmapsto G$

such that:

1) $\forall\, a, b, c \in M : \; a \cdot (b \cdot c) = (a \cdot b) \cdot c$

2) $\exists\, 1_G \in M \;/\; \forall\, a \in M : 1_G \cdot a = a \cdot 1_G = a$

3) $\forall\, a \in M \;\exists\, a^{-1} \in M \;/\; a \cdot a^{-1} = a^{-1} \cdot a = 1$

Consider a group $G = (M, \cdot)$ such that

4) $\forall\, a, b \in M \quad a \cdot b = b \cdot a$

then the group is said to be "abelian".

Examples:

1) $G = (\mathbb{Z}, +)$ whereas: $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ and "+" = usual sum.

$a, b \in \mathbb{Z} \quad a + b \in \mathbb{Z}$

Then:

1) $(a+b)+c = a+(b+c)$ OK

2) $1_G = 0$. Infact: $a + 0 = 0 + a \quad \forall\, a \in \mathbb{Z}$

3) $a^{-1} = -a$. In fact: $a+(-a) = (-a) + a = 0$

$(2+1)+5 = 2+(1$

$2+0 = 0+2 = 2$

$2-2 = 0$

Moreover, this group is also abelian, being $a+b = b+a$.

ii) $(N, +)$ where $N = \{1, 2, 3, \dots\}$

is <u>not</u> a group.

$+$ is an internal operation but the "unity" (ie the zero)
is not part of N. Also the inverse is not.

(iii) $(R, +)$ is a group (abelian) where: R = real numbers
$+$ = sum

(iv) what about $(R, \cdot)$, where $\cdot$ = usual number multiplication

$\cdot : R \times R \mapsto R$ is fullfilled.

1) $(a \cdot b) \cdot c = a(b \cdot c)$ is ok.

2) $1_G = 1$ ~~was~~ ok... $\forall a \in R \quad a \cdot 1 = 1 \cdot a = a \qquad (3 \cdot 1 = 3)$

3) inverse:
$\forall a \; \exists \frac{1}{a} \; / \; a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1 \qquad (3 \cdot \frac{1}{3} = \frac{1}{3} \cdot 3 = 1)$

But here there is a problem: $a = 0$ has <u>no</u> inverse....

that is: $(R, \cdot)$ is not a group.

(v) $(R - \{0\}, \cdot)$ is an abelian group because also the third property is fullfilled.

Vector space E on the "body" of the real numbers:

$\exists$ operation $+$ such that $(E, +)$ is an abelian group

$\exists$ an operation $\cdot : R \times E \mapsto E$ /

$a \cdot (b \cdot \vec{u}) = (a \cdot b) \cdot \vec{u}$ $\quad \forall a, b \in R$ and $\forall \vec{u} \in E$

$(a+b) \cdot \vec{u} = a\vec{u} + b\vec{u}$ $\quad \forall a, b \in R$ and $\forall \vec{u}$

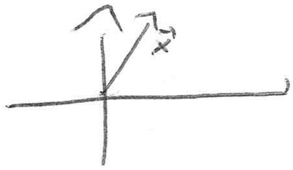
$a \cdot (\vec{u} + \vec{v}) = a\vec{u} + b\vec{v}$ $\quad \forall a \in R$ and $\forall \vec{u}, \vec{v} \in E$.

Moreover, a vector space E is normed if there we define an operation

$\| \cdot \| = E \times E \mapsto R^+ + \{0\}$ /

$$\begin{cases} \|\vec{u}\| \geq 0 \text{ and } \|\vec{u}\| = 0 \iff \vec{u} = \vec{0} \\ \|a \cdot \vec{u}\| = |a| \, \|\vec{u}\| \\ \|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\| \end{cases}$$

nb: E is also a metric vector space with $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$



<u>Example</u>:

$E = R^2$ is a vector space. $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$(E, +)$ is a group $\Rightarrow$ (abelian group, indeed) simple to prove.

$a \cdot \vec{x} = \begin{pmatrix} ax \\ ay \end{pmatrix}$ fullfills all the required

$\|\vec{x}\| = \sqrt{x^2 + y^2}$ makes R^2 a "normed" space.

Let us now consider 2x2 matrices of real numbers.

Example:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$ with $a, b, c, d \in R$

$G = (A, +)$ is an abelian group, whereas $+$:

$$A + B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$$

In general, we can consider $N \times N$ real square matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & \cdots & & a_{2N} \\ \vdots & & & \\ a_{N1} & \cdots & & a_{NN} \end{pmatrix}$$

(They also form an abelian group under the sum)

A 2x2 matrix A can be seen as an operator on R_2 space:

$$A \cdot \vec{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} x + a_{12} y \\ a_{21} x + a_{22} y \end{pmatrix}$$

$\vec{x} \in R^2$

This operator $A : R^2 \mapsto R^2$ is linear:

$$A(\alpha \vec{x} + \beta \vec{y}) = \alpha (A \vec{x}) + \beta (A \vec{y}) \qquad \forall \vec{x}, \vec{y} \in R^2$$

and

$$\forall \alpha, \beta \in R.$$

Similarly, a $N \times N$ matrix A can be regarded as a linear operator on the vector space R^N:

$$A : R^N \mapsto R^N.$$

Trace

$$\mathrm{Tr}\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right] = a_{11} + a_{22} = \sum_{i=1}^{N=2} a_{ii}$$

In general for a $N \times N$ matrix:

$$\mathrm{Tr}\,A = \sum_{i=1}^{N} a_{ii}$$

Properties:

$$\mathrm{Tr}[A+B] = \mathrm{tr}[A] + \mathrm{tr}[B]$$

$$\mathrm{Tr}[c \cdot A] = c\,\mathrm{Tr}[A]$$

$$\mathrm{Tr}[A] = \mathrm{Tr}[A^t]$$

(Achtung: $\mathrm{Tr}[AB] \neq \mathrm{Tr}[A] \cdot \mathrm{Tr}[B$

$$\mathrm{Tr}[A \cdot B] = \mathrm{Tr}[B \cdot A]$$

$$\mathrm{Tr}[ABC] = \mathrm{Tr}[CAB] = \mathrm{Tr}[BCA]$$

$$\mathrm{Tr}[1_N] = N.$$

Determinant

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det A = a_{11} a_{22} - a_{12} a_{21} = \sum_{i,j=1}^{2} a_{1i} a_{2j} \varepsilon_{ij}$$

whereas $\varepsilon_{ij} /$ $\varepsilon_{12} = 1$, $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{21} = -1$

In fact

$$\varepsilon_{12} a_{11} a_{22} + \varepsilon_{21} a_{12} a_{21} = a_{11} a_{22} - a_{12} a_{21} = \det A.$$

In general, for A being an $N \times N$ matrix:

$$\boxed{\det A = \sum_{i_1, \ldots, i_N = 1}^{N} \varepsilon_{i_1 i_2 \ldots i_N} a_{1 i_1} a_{2 i_2} \cdots a_{N i_N}}$$

$$\varepsilon_{i_1 \ldots i_N} = \begin{cases} 0 & \text{if two indices are equals} \\ 1 & \text{even permutation} \\ -1 & \text{odd} \quad " \quad " \end{cases}$$

Properties of the determinant:

$$\det(c \cdot A) = c^N \det A$$

$$\det(A^t) = \det A$$

$$\det(A \cdot B) = \det A \cdot \det B$$

Achtung:
$(\det(A+B) \neq \det A + \det B)$

$$\det 1_N = 1$$

Inverse matrix

A is a $N \times N$ matrix.

A^{-1} is the inverse matrix of A if

$$A \cdot A^{-1} = A^{-1} \cdot A = 1_N$$

$$\boxed{A^{-1} \text{ exist only if } \det A \neq 0 \,!}$$

For instance for a 2×2 matrix:

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

- The inversion of matrix is an operation which has to be performed when solving complicated problems.

- The set of $N \times N$ matrices is not a group under the matrix operation because not all the elements have the inverse.

- The set of $N \times N$ matrices such that $\det A \neq 0$ form a group (a nonabelian one) under the matrix multiplication.