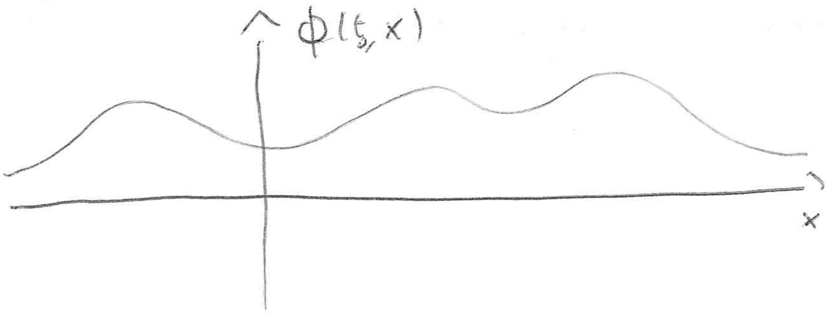
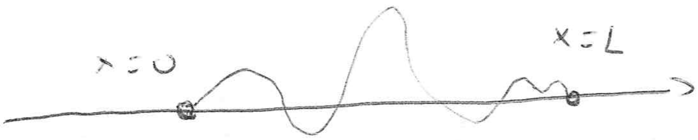
Solution in the "guitar case"

- $\boxed{t_5}$



$\phi(t,x) = h$: height at a given "t" and a given x".

An interesting case is that of a guitar string.



$$\begin{cases} \phi(t,0) = 0 & \forall t \\ \phi(t,L) = 0 & \forall t \end{cases}$$

We search now solutions which fullfill these boundary conditions.

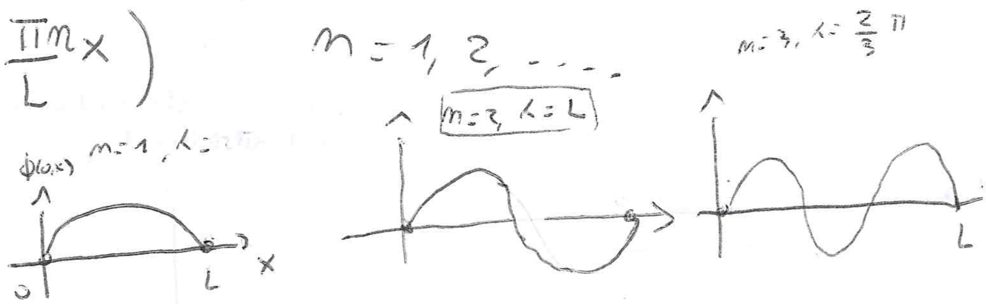
In particular, let us start from the case $t = 0$:

$\phi(0,x)$ / $\phi(0,0) = \phi(0,L) = 0$.

so dass

$$\phi_m(0,x) = \sin\left(\frac{\pi m}{L} x\right) \qquad m = 1, 2, \dots$$

In fact:



$$\phi_m(0,0) = 0$$

$$\phi_m(0,L) = \sin\left(\frac{\pi m}{L} L\right) = \sin(\pi m) = 0 \quad ✓$$

$$\boxed{\lambda = \frac{2L}{m}}$$

N.B:

$m=0 \rightarrow \phi_0(0,x) = 0$ trivial

$m=-1 \rightarrow \phi_{-1}(0,x) = -\phi_1(0,x)$ NOT independent

...

This is why we consider only $m = 1, 2, \dots$

Now, we could write the solution

$$\phi_m(t,x) = \sin\left(\frac{\pi m}{L}(x - ct)\right)$$

This is a solution of the diff. eq. but the boundary conditions are broken:

$$\phi_m(t,0) = \sin\left(\frac{\pi m}{L}(-ct)\right) \neq 0 \quad \text{in general}\dots$$

$$\phi_m(t,L) = \sin\left(\frac{\pi m}{L}(L - ct)\right) \neq 0 \quad \text{" " " "}$$

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We can obtain the desired solution by subtracting a piece with $x+ct$.

$$\phi_n(t,x) = \frac{1}{2}\left(\sin\left(\frac{\pi n}{L}(x-ct)\right) + \sin\left(\frac{\pi n}{L}(x+ct)\right)\right)$$

$$\phi_n(0,x) = \sin\left(\frac{\pi n}{L}x\right)$$ as before... but:

$$\phi_n(t,0) = \frac{1}{2}\left(\sin\left(\frac{\pi n}{L}(-ct)\right) + \sin\left(\frac{\pi n}{L}ct\right)\right) = 0$$

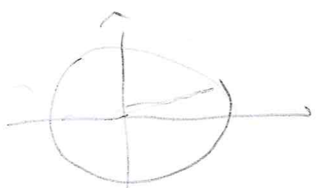
$$\forall t$$

and also

$$\phi_n(t,L) = \frac{1}{2}\left(\sin\left(\frac{\pi n}{L}(L-ct)\right) + \sin\left(\frac{\pi n}{L}(L+ct)\right)\right)$$

$$= \frac{1}{2}\left(\sin\left(\pi n - \frac{\pi n}{L}ct\right) + \sin\left(\pi n + \frac{\pi n}{L}ct\right)\right)$$

$$= \frac{1}{2}\left((-1)^n \sin\left(-\frac{\pi n}{L}ct\right) + (-1)^n \sin\left(\frac{\pi n}{L}ct\right)\right) = 0!$$



A general solution of the diff. eq. fulfilling the boundary condition is given by:

$$\phi(t,x) = \sum_{n=1}^{\infty} c_n \, \phi_n(t,x)$$

$$\phi_n(t,x) = \frac{1}{2}\left(\sin\left(\frac{\pi n}{L}(x-ct)\right) + \sin\left(\frac{\pi n}{L}(x+ct)\right)\right)$$

This is a particular example of a "Fourier decomposition.

We have written the solution as a "sum of harmonics"...

the "ground state" $(n=1)$ and the higher frequencies.

The difference of instruments relies exactly on the different intensity of $c_{n>1}$ ...

$$\begin{cases} |c_1| & \text{"larger coefficient"} \\ c_2, c_3, \ldots & \text{higher harmonics} \end{cases}$$