Raphael Bombelli 1572

Georolamo Cardano 1545

"i" → imaginary ...

↳ "strange name" for a mathematical object...
(remember the irrational nrs as $\sqrt{2}$?)

$i \cdot i = -1$.

$x^2 = -1 \Rightarrow$ no real solution.

$z^2 = -1 \Rightarrow \begin{cases} z = i \\ z = -i \end{cases}$ are then a solution (by definition!!!)
still, one regarded this "i" as a strange beast ...

Formally: $\boxed{i = \sqrt{-1}}$

In general, a complex nr is then given by two pieces:

$z = x + iy$ $\quad x, y \in \mathbb{R}$

$z \in \mathbb{C}$.

$\begin{cases} \mathrm{Re}\, z = x \\ \mathrm{Im}\, z = y \end{cases}$

Note that:
$\begin{cases} 2^{\circ} \text{ order eq: } & ax^2 + bx + y = 0 \Rightarrow \text{ the solutions are complex if } \Delta = b^2 - 4ac < 0 !! \\ \text{Cardano} \to \text{solution of eq. of order "3"}. \end{cases}$

⇓

$z^2 = -3$

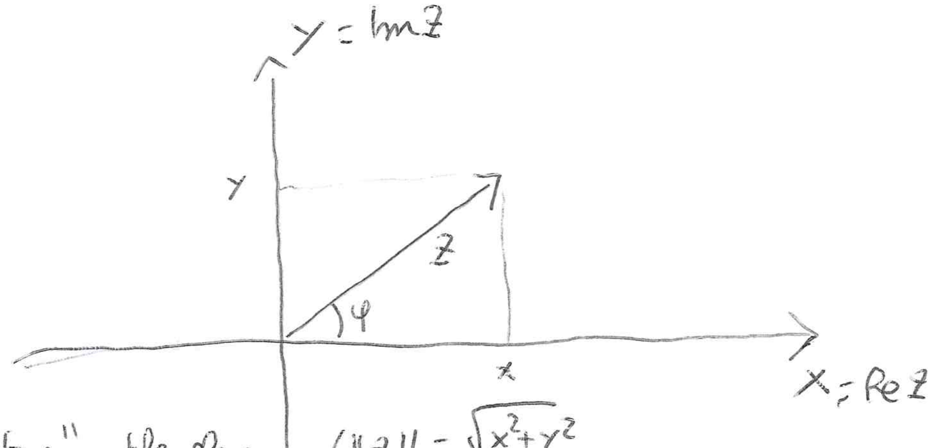
$\boxed{z = \pm i\sqrt{3}}$.

How to visualize complex nr ?

This has been an important step toward their acceptance!

<u>Wessel</u>'s complex plane (Caspar Wessel 1797)

$\hookrightarrow$ 200 years later !!!!!



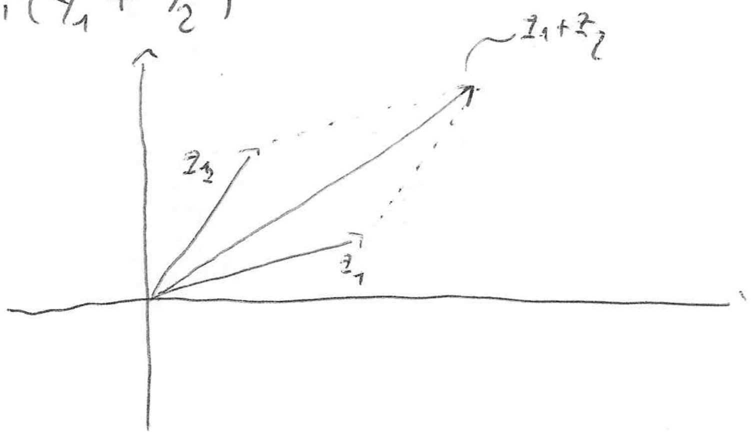
z is then a "vector" in the plane.

$$\begin{cases} \|z\| = \sqrt{x^2+y^2} \\ \tan\varphi = \dfrac{y}{x} \end{cases}$$

$z_1 = x_1 + i y_1$

$z_2 = x_2 + i y_2$

$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$



$z_1 + z_2$ "just as the sum of two vectors" ...

Then, the question is:

"Why complex nrs?"

"Why not to work directly with vectors"?

The reason is: we can now build a multiplication of complex nrs:

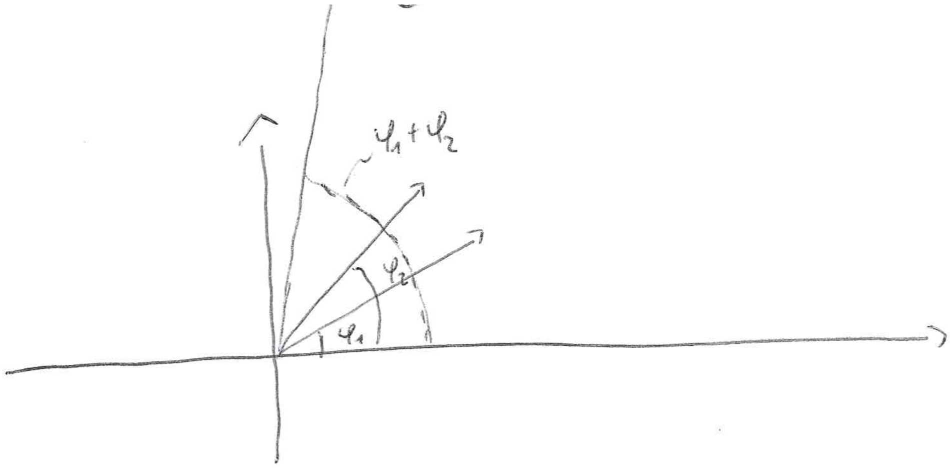
$$\begin{cases} z_1 = x_1 + i y_1 \\ z_2 = x_2 + i y_2 \end{cases}$$

$$z_1 z_2 = (x_1 + i y_1)(x_2 + i y_2) = x_1 x_2 + \underbrace{i^2}_{-1} y_1 y_2 + i(y_1 x_2 + x_1 y_2)$$

$$= x_1 x_2 - y_1 y_2 + i(x_1 y_2 + y_1 x_2)$$

$$\begin{cases} Re(z_1 z_2) = x_1 x_2 - y_1 y_2 \\ Im(z_1 z_2) = x_1 y_2 + y_1 x_2 \end{cases}$$

What does $z_1 z_2$ represents graphically?



$$\|z_1 z_2\| = \sqrt{(x_1x_2 - y_1y_2)^2 + (x_1y_2 + y_1x_2)^2} =$$

$$= \sqrt{x_1^2x_2^2 + y_1^2y_2^2 \cancel{-2x_1x_2y_1y_2} + x_1^2y_2^2 + y_1^2x_2^2 \cancel{+2x_1y_2y_1x_2}}$$

$$= \sqrt{x_1^2(x_2^2+y_2^2) + y_1^2(x_2^2+y_2^2)}$$

$$= \sqrt{(x_1^2+y_1^2)(x_2^2+y_2^2)} =$$

$$= \sqrt{x_1^2+y_1^2}\,\sqrt{x_2^2+y_2^2} = \|z_1\|\,\|z_2\| \quad \text{q.e.d.}$$

e^{ix}

What is e^{ix}? Remind that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$; then...

$$e^{ix} = \sum_{m=0}^{\infty} \frac{1}{m!} (ix)^m = \sum_{m=0}^{\infty} \frac{(i)^n}{m!} x^m$$

$i^2 = -1$

$i^3 = -i$

$i^4 = +1$ ...

$$\dots \begin{cases} (i)^{2m} = (-1)^m \\ (i)^{2m+1} = (-1)^m i \end{cases}$$

Then we divide even and odd powers:

$$e^{ix} = \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m}}_{\cos(x)} + i \underbrace{\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1}}_{\sin x}$$

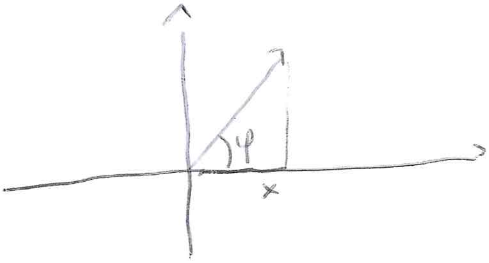
$$= \cos x + i \sin x$$

$$\boxed{e^{ix} = \cos x + i \sin x}$$

$$x = \pi \rightarrow e^{i\pi} = \underbrace{\cos \pi}_{-1} + i \underbrace{\sin \pi}_{=0} = -1 \Rightarrow \boxed{e^{i\pi} + 1 = 0}$$

Mystical equation

Polar representation of complex nrs



$z = x + iy$

$\|z\| = \sqrt{x^2+y^2} = \rho$ "rho"

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

Ergo:

$$z = \rho\cos\varphi + i\rho\sin\varphi = \rho(\cos\varphi + i\sin\varphi) = \rho e^{i\varphi}.$$

$$\boxed{z = \underbrace{\rho}_{\text{length}} e^{i\varphi}}$$

angle

The product of complex nrs can be now seen in a very simple way:

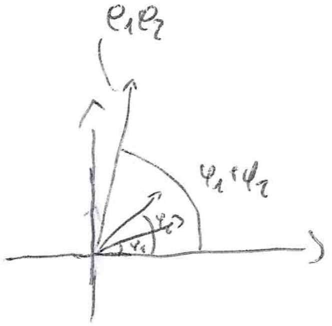
$z_1 = \rho_1 e^{i\varphi_1}$

$z_2 = \rho_2 e^{i\varphi_2}$

$$z_1 z_2 = \rho_1 e^{i\varphi_1}\rho_2 e^{i\varphi_2} = (\rho_1 + \rho_2)e^{i(\varphi_1+\varphi_2)}$$

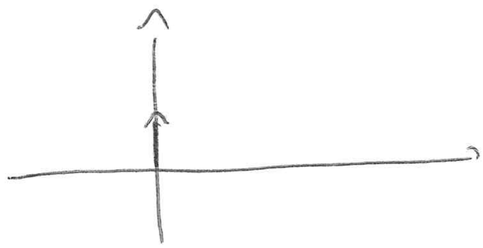
Ergo we see that:

$$\begin{cases} \|z_1 z_2\| = \rho_1 \rho_2 \\ \text{Angle}(z_1 z_2) = \varphi_1 + \varphi_2 \end{cases}$$



Then, we also immediately see why $i^2 = -1$!!!

In fact:



Write i as:

$$i = e^{i\pi/2} = \underbrace{\cos(\pi/2)}_{0} + i\underbrace{\sin(\pi/2)}_{1} = i.$$

$$i \cdot i = e^{i\pi/2} e^{i\pi/2} = e^{i\pi} = -1 \;!!$$

Length of i: $\|i\| = 1$. $\text{Angle}(i) = \pi/2$.

$i \cdot i \longrightarrow$ length still "1"

Angle $\longrightarrow \frac{\pi}{2} + \frac{\pi}{2} = \pi$!!!!

All what we could do with real nrs (sum, multipl., ...

body

can be also done with complex nrs.

But there is now something more

"Fundamental theorem of Algebra"

Eq. of n-th order in z

$$a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n = 0$$

(with $a_i \in \mathbb{C}$)

always admit n solutions !!! $z_1, z_2, \dots, z_n$!

$$a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n =$$

$$= a_n (z - z_1)(z - z_2) \dots (z - z_n)$$

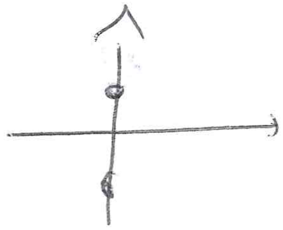
(Eventually, some of the z_i can be equal....)

This does not so with real nrs...

For instance: $x^2+1=0$

no solution for $x \in R$

$z^2+1=0 \rightarrow z = \pm i$



$z^2+1 = (z+i)(z-i)$

What are complex nrs good for?

⇝ Solution of (real) diff. eq.

$$y'' + \eta y = 0 \quad \begin{cases} \eta > 0 & \cos(\sqrt{\eta}\,x),\ \sin(\sqrt{\eta}\,x) \\ \eta < 0 & e^{\sqrt{-\eta}\,x},\ e^{-\sqrt{-\eta}\,x} \end{cases}$$

But now we can simply write down the most general solution as:

$$y(x) = A e^{\sqrt{-\eta}\,x} + B e^{-\sqrt{-\eta}\,x} \qquad (A, B \in \mathbb{C})$$

also for $\eta > 0$ ... In fact, for $\eta > 0$ we simply have $\sqrt{-\eta} = i\sqrt{\eta}$ and we go back to $\cos(\sqrt{\eta}\,x)$ and $\sin(\sqrt{\eta}\,x)$

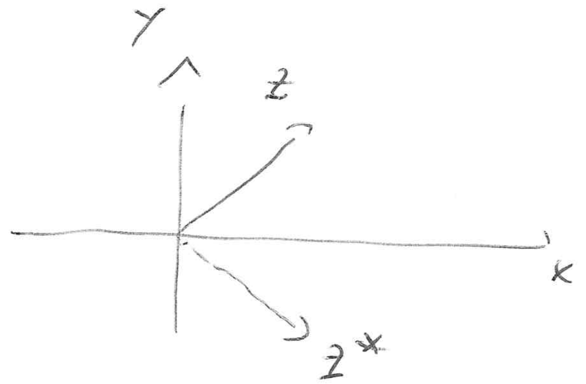
But now A, B are complex and $y(x) : \mathbb{R} \mapsto \mathbb{C}$ (in general)

How to make sure to have real solution?

Complex conjugation:

$z = x + iy$

$z^* = x - iy$



if now $z = z^*$

$x + iy = x - iy \implies y = 0$

then $z = x$ is real...

we can then impose that

$y^*(x) = y(x)$

to make sure to keep a real solution.

$\implies$ Electromagnetism $\rightarrow$ Complex solutions, but indeed only "real ones" are physical

$\implies$ QM $\rightarrow$ "i" from the very beginning there