Aufgabenblatt 1 : Lösungen

1)

1.1) $f(x) = x^n$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$(x+h)^n = x^n + n x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \binom{n}{3} x^{n-3} h^3 + \dots + h^n$$

Ergo:

$$\lim_{h \to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h \to 0} \frac{\cancel{x^n} + n x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots \cancel{- x^n}}{h} =$$

$$= \lim_{h \to 0} \frac{n x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots}{h} = \lim_{h \to 0} \left(n x^{n-1} + \binom{n}{2} x^{n-2} h + \dots \right)$$

$$= n x^{n-1} \quad \text{q.e.d.}$$

2)

$$\frac{d}{dx}\left(f(x) g(x)\right) = \lim_{h \to 0} \frac{f(x+h) g(x+h) - f(x) g(x)}{h}$$

For h very small we have:

$$f(x+h) = f(x) + h f'(x)$$

$$g(x+h) = g(x) + h g'(x)$$

Ergo:

$$\lim_{h \to 0} \frac{\big(f(x) + h f'(x)\big)\big(g(x) + h g'(x)\big) - f(x) g(x)}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{fg} + h f' g + h f g' + h^2 f' g' - \cancel{fg}}{h} =$$

$$= \lim_{h \to 0} \left(f' g + f g' + h f' g' \right) = f' g + f g' = f'(x) g(x) + f(x) g'(x)$$

q.e.d.

1.3)

$$f(x) = \frac{\ln(1+ax^2)}{x}$$

$$\lim_{x \mapsto 0} \frac{\ln(1+ax^2)}{x} = \left[\frac{0}{0}\right]$$

Hospital:

$$\lim_{x \mapsto 0} \frac{2ax \cdot \frac{1}{1+ax^2}}{1} = \lim_{x \mapsto 0} \frac{2ax}{1+ax^2} = 0.$$

1.4).

$$f(x) = \ln\left(1+e^{-\sqrt{x^2+a^2}}\right)$$

$$f'(x) = \left(\frac{d}{dx}\left(1+e^{-\sqrt{x^2+a^2}}\right)\right)\frac{1}{1+e^{-\sqrt{x^2+a^2}}} = \left(\frac{d}{dx}\left(e^{-\sqrt{x^2+a^2}}\right)\right)\cdot\frac{1}{1+e^{-\sqrt{x^2+a^2}}} =$$

$$= \left(-\frac{d}{dx}\sqrt{x^2+a^2}\right)e^{-\sqrt{x^2+a^2}}\frac{1}{1+e^{-\sqrt{x^2+a^2}}} = -\cancel{2}x\cdot\frac{1}{\cancel{2}\sqrt{x^2+a^2}}\cdot\frac{e^{-\sqrt{x^2+a^2}}}{1+e^{-\sqrt{x^2+a^2}}} =$$

$$\boxed{f'(x) = -\frac{x}{\sqrt{x^2+a^2}}\frac{1}{e^{\sqrt{x^2+a^2}}-1}}$$

1.5)

$$f(x) = e^{-1/x^2}$$

$$f'(x) = \left(-\frac{d}{dx}\left(\frac{1}{x^2}\right)\right) e^{-1/x^2} = 2x^{-3} e^{-1/x^2}$$

$$\lim_{x \mapsto 0} f'(x) = \lim_{x \mapsto 0} \frac{2x^{-3}}{e^{1/x^2}} \overset{\text{Hoptal}}{=} \lim_{x \mapsto 0} \frac{-6x^{-4}}{-2x^{-3} e^{1/x^2}}$$

$$= \lim_{x \mapsto 0} \frac{3x^{-1}}{e^{1/x^2}} = \lim_{x \mapsto 0} \frac{-3x^{-2}}{-2x^{-3} e^{1/x^2}} =$$

$$= \lim_{x \mapsto 0} \frac{3x}{2e^{1/x^2}} = \frac{0}{\infty} = 0 \,!$$

$$\boxed{\lim_{x \mapsto 0} f'(x) = 0.}$$

2)

2.1) $D = \mathbb{R}$ (the function is well-defined allover $\mathbb{R}$)

2.2) $\lim\limits_{x \mapsto +\infty} x e^{-x^2} = \lim\limits_{x \mapsto \infty} \dfrac{x}{e^{x^2}} = \lim\limits_{x \mapsto \infty} \dfrac{1}{2x e^{x^2}} = \dfrac{1}{\infty} = 0.$

$\lim\limits_{x \mapsto -\infty} x e^{-x^2} = 0.$

$x e^{-x^2} > 0$ for $x > 0$

$x e^{-x^2} < 0$ for $x < 0$

$x e^{-x^2} = 0$ for $x = 0$

$f'(x) = e^{-x^2} + x(-2x)e^{-x^2} = (1-2x^2)e^{-x^2} = 0$

$\Downarrow$

$x = \pm \dfrac{1}{\sqrt{2}}$

$f''(x) = (-4x)e^{-x^2} + (1-2x^2)(-2x)e^{-x^2} =$

$= (-4x - 2x + 4x^2)e^{-x^2}$

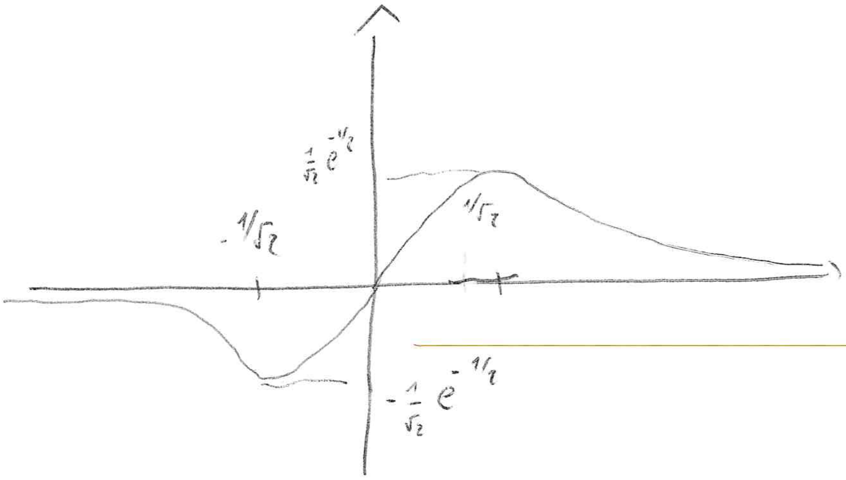
$= (4x^2 - 6x)e^{-x^2}$

$f''(\frac{1}{\sqrt{2}}) = \left(\frac{4}{2} - \frac{6}{\sqrt{2}}\right)e^{-1/2} < 0$; $f''(-\frac{1}{\sqrt{2}}) = \left(\frac{4}{2} + \frac{6}{\sqrt{2}}\right)e^{-1/2} > 0.$

Also:

$x = \frac{1}{\sqrt{2}} \mapsto$ corresponds to a maximum.

$x = -\frac{1}{\sqrt{2}} \mapsto$ " " " " minimum.



2) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$; $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$

$x e^{-x^2} = \sum_{n=0}^{\infty} \frac{x(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{-x \cdot x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!} =$

$= \frac{x}{0!} - \frac{x^3}{1!} + \frac{x^5}{2!} - \frac{x^7}{3!} + O_9(x^8).$

$$\boxed{= x - x^3 + \frac{1}{2}x^5 - \frac{1}{6}x^7 + O_9(x^8)}$$

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2.3) $f(x) = x e^{-x^2}$

$f(1) = \frac{1}{e}$

$f'(x) = (1-2x^2)e^{-x^2} \rightarrow f'(1) = \frac{(1-2)}{e} = -\frac{1}{e}$

$f''(x) = (4x^2-6x)e^{-x^2} \rightarrow f''(1) = \frac{(4-6)}{e} = -\frac{2}{e}$

Ergo:

$$f(x) \simeq \frac{1}{e} - \frac{1}{e}(x-1) - \frac{1}{2}\frac{2}{e}(x-1)^2 + O_g((x-1)^3)$$

2.4)

$F(x) = -\frac{1}{2} e^{-x^2}$

In fact:

$\frac{dF}{dx} = -\frac{1}{2}(-2x)e^{-x^2} = xe^{-x^2}$.

$$\int_{-1}^{1} f(x)dx = \left(-\frac{1}{2}e^{-x^2}\right)_{-1}^{1} = -\frac{1}{2}\left(e^{-1} - e^{-1}\right) = 0 \,!$$

(This is intuitive, because the function is odd and the integral zero)

$$\int_0^1 f(x)dx = \left(-\frac{1}{2}e^{-x^2}\right)_0^1 = -\frac{1}{2}\frac{1}{e} - \left(-\frac{1}{2}\right) = \frac{1}{2}\left(1-\frac{1}{e}\right) = \frac{1}{2}\frac{e-1}{e} > 0$$