## Ex 1.

1.1 • $\int_{-\infty}^{\infty} x^2 \delta(x)\, dx = 0^2 = 0$

• $\delta(2x+1) = \delta\left(2\left(x+\frac{1}{2}\right)\right) = \frac{1}{2}\delta\left(x+\frac{1}{2}\right)$

Ergo:

$$\int_{-\infty}^{\infty} x^2 \delta(2x+1)dx = \int_{-\infty}^{\infty} x^2 \frac{1}{2}\delta\left(x+\frac{1}{2}\right) = \frac{1}{2}\left(-\frac{1}{2}\right)^2 = \frac{1}{8}.$$

• $\int_{-1}^{5} dx\, f(x+1)\,\delta(x) = f(1)$ $\quad (x=0 \dots)$

1.2 $\int_{0}^{\infty} \delta(x) dx$

Let us study the $\delta_\varepsilon(x)$-representation:

$$\delta_\varepsilon(x) = \begin{cases} 0 & |x| > \varepsilon \\ \frac{1}{2\varepsilon} & |x| \leq \varepsilon \end{cases}$$

$$\int_0^\infty \delta_\varepsilon(x) dx = \int_0^\varepsilon \frac{1}{2\varepsilon} dx = \frac{1}{2} \quad (\text{indep. on } \varepsilon).$$

Therefore:

$$\int_0^\infty \lim_{\varepsilon \to 0} \delta_\varepsilon(x)\, dx = \int_0^\infty \delta(x) dx = \frac{1}{2}.$$

1.3) $\delta(x^2+x)$

$$f(x)=x^2+x = x(x+1) = 0 \begin{cases} x=0 \\ x=-1 \end{cases}$$

$$f'(x) = 2x+1$$

$$f'(0) = 1$$

$$f'(-1) = 2(-1)+1 = -1$$

Ergo:

$$\delta(x^2+x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x+1)$$

$$\int_{-\infty}^{\infty} dx\,(x^7+x^6)\,\delta(x^2+x) = \int_{-\infty}^{\infty} dx\,(x^7+x^6)\left(\frac{1}{2}\delta(x) + \frac{1}{2}\delta(x+1)\right) =$$

$$= \frac{1}{2}(0+0) + \frac{1}{2}\left((-1)^7 + (-1)^6\right) = \frac{1}{2}\cdot 0 + \frac{1}{2}(-1+1) = 0+0=0$$

1.4) $\int_{-\infty}^{\infty} dx\, \theta(x)\, \delta(x^2-1)$

Same procedure:

$$\delta(x^2-1) = \frac{1}{2}\delta(x+1) + \frac{1}{2}\delta(x-1)$$

$$\int_{-\infty}^{\infty} dx\, \theta(x)\left[\frac{1}{2}\delta(x+1) + \frac{1}{2}\delta(x-1)\right] = \frac{1}{2}\theta(-1) + \frac{1}{2}\theta(1) =$$

$$= \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = \frac{1}{2}.$$

1.5

$$\int_{-\infty}^{\infty} dx \, \delta(x^2+1)$$

$f(x) = x^2+1 = 0 \quad \rightarrow$ Never!

Ergo:

$\delta(x^2+1) = 0 \quad \forall x \in R.$

Therefore:

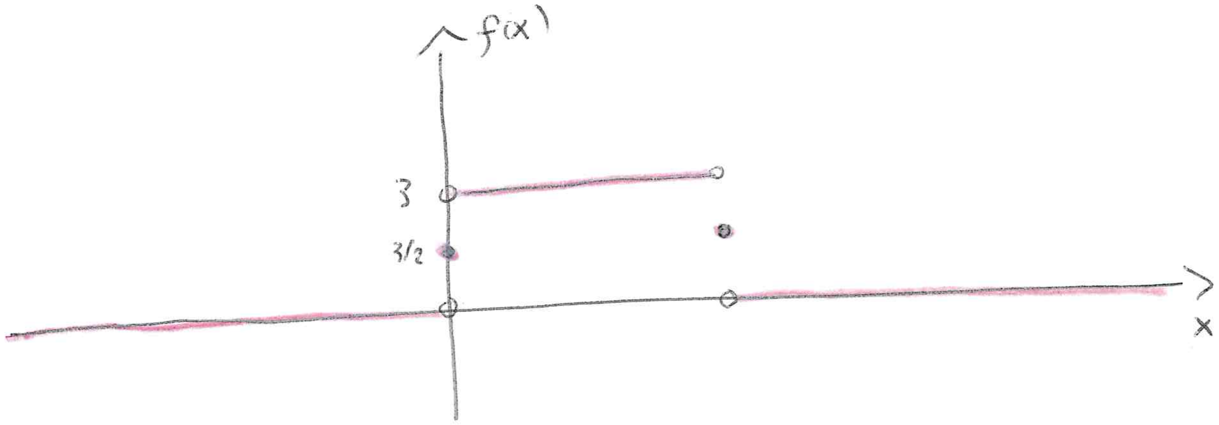
$$\int_{-\infty}^{\infty} dx \, \delta(x^2+1) = 0.$$

2)

2.1 $f(x) = 3\,\vartheta(x)\,\vartheta(L-x)$

(i) $L > 0$

$$f(x) = \begin{cases} 0 & x < 0 \\ 3/2 & x = 0 \\ 3 & 0 < x < L \\ 3/2 & x = L \\ 0 & x > L \end{cases}$$




$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 3L}$$

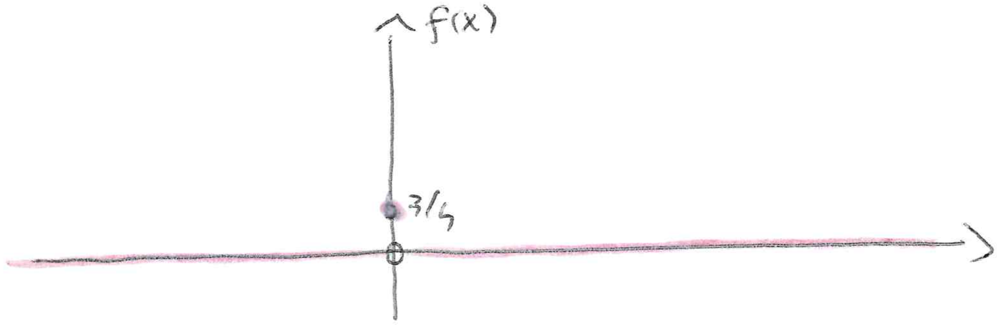
$$f'(x) = 3\,\delta(x)\,\vartheta(L-x) + 3\,\vartheta(x)(-1)\,\delta(L-x)$$

$$= 3\,\delta(x)\,\vartheta(L) - 3\,\vartheta(L)\,\delta(x-L)$$

$$\boxed{f'(x) = 3\left(\delta(x) - \delta(x-L)\right)}$$

(ii) $L = 0$

$$f(x) = 3\theta(x)\,\theta(-x) = \begin{cases} 0 & \forall\, x \neq 0 \\ \frac{3}{4} & \text{for } x = 0 \end{cases}$$




$$\int_{-\infty}^{\infty} f(x)\,dx = 0$$

(one can also see it as the limit $L \mapsto 0$ of the case (i)).

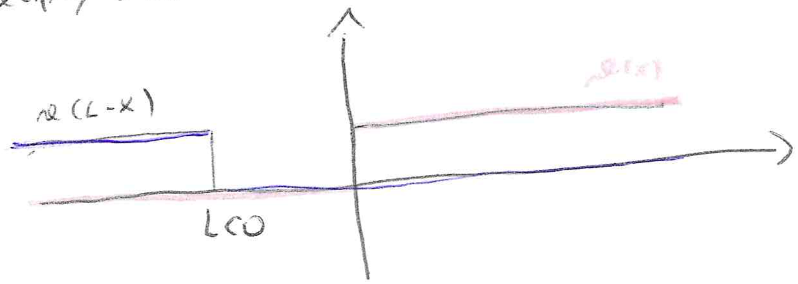
$f'(x) = 0 \quad \forall\, x \neq 0.$

This function is then "strictly speaking" not derivable in $x = 0$.

(iii) $L < 0$

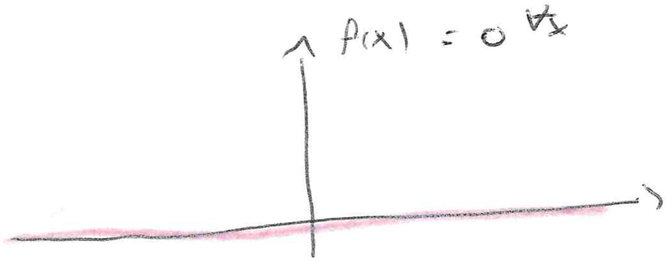
$$f(x) = 3\,\theta(x)\,\theta(L-x) = 0 \quad \forall x.$$

In fact I multiply these two functions:




Either one or the other or both are zero.

Ergo:



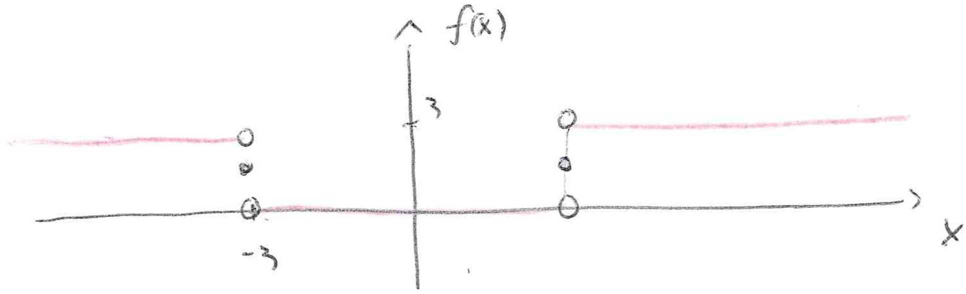

$$\int_{-\infty}^{\infty} f(x)\,dx = 0$$

$$\frac{df}{dx} = 0 \quad \forall x.$$

2.2) $f(x) = 3\,u(3x^2 - 27)$

$3x^2 - 27 > 0 \rightarrow x^2 > 9 \rightarrow |x| > 3$

Ergo:




$$f(x) = \begin{cases} 3 & |x| > 3 \\ 3/2 & x = -3 \text{ and } x = +3 \\ 0 & |x| < 3 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = +\infty .$$

$$\frac{d f(x)}{dx} = 3 \cdot 6x \;\delta(3x^2 - 27) = 12x\;\delta(3x^2-27)$$

$f(x) = 3x^2 - 27 = 0 \quad x = \pm 3$

$f'(x) = 6x \begin{cases} 18 \\ -18 \end{cases}$

Ergo:

$$\frac{df}{dx} = 18x \left[ \frac{1}{18}\,\delta(x-3) + \frac{1}{18}\,\delta(x+3) \right] = x\,\delta(x-3) + x\,\delta(x+3)$$

$$= 3\,\delta(x-3) - 3\,\delta(x+3)$$

Note that we could have expressed $\theta(x)$ as:

$$f(x) = 3\,\theta(-3-x) + 3\,\theta(x-3)$$

upon deriving:

$$f'(x) = 3(-1)\,\delta(-3-x) + 3\,\delta(x-3)$$

$$= 3\,\delta(x-3) - 3\,\delta(x+3)$$

(same result)