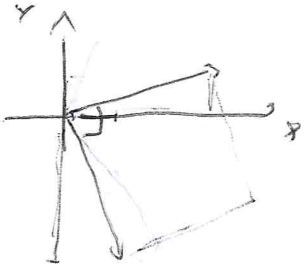
1.1

$$\vec{x}_1 \cdot \vec{x}_2 = x_1 x_2 + y_1 y_2$$

First, note that

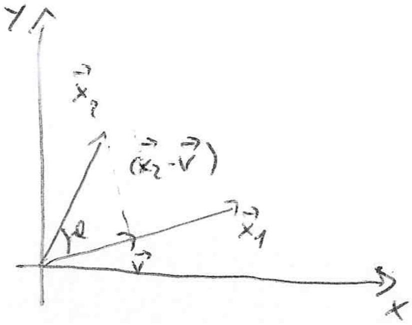
$$\vec{x}_1 \cdot \vec{x}_2 = 0 \quad \text{if } \alpha = 90^\circ$$



In fact:

$\vec{x}_2 = \lambda (y_1, -x_1)$ in this case.

Note now:



$\vec{v} \parallel \vec{x}_1$. $\vec{v}$ is parallel to $\vec{x}_1$.

$\vec{v} = \alpha (x_1, y_1)$. $\quad \|\vec{v}\| = \alpha \cdot \sqrt{x_1^2 + y_1^2} = \|\vec{x}_2\| \cos\alpha$

$(\vec{x}_2 - \vec{v})$ is $\perp$ to $\vec{x}_1$. Ergo

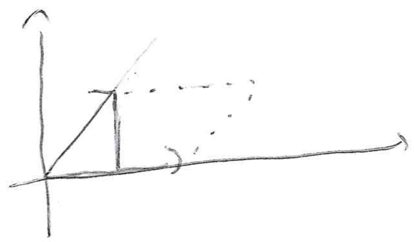
$$(x_2 - \alpha x_1, y_2 - \alpha y_1)(x_1, y_1) = x_1 x_2 - \alpha x_1^2 + y_1 y_2 - \alpha y_1^2 = 0$$

$$\alpha = \frac{x_1 x_2 + y_1 y_2}{x_1^2 + y_1^2}$$

Ergo:

$$\cos\alpha = \alpha \frac{\sqrt{x_1^2 + y_1^2}}{\sqrt{x_2^2 + y_2^2}} = \frac{x_1 x_2 + y_1 y_2}{\sqrt{x_1^2 + y_1^2}\,\sqrt{x_2^2 + y_2^2}} \Rightarrow \boxed{x_1 x_2 + y_1 y_2 = |\vec{x}_1| |\vec{x}_2| \cos\alpha \quad \text{q.e.d.}}$$

Simpler proof : $\vec{x}_1 = (x_1, 0)$. This is actually no loss of generality ...

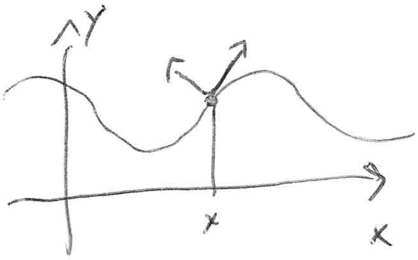


$$\vec{x}_1 \vec{x}_2 = x_1 x_2$$

But $x_2 = |\vec{x}_2| \cdot \cos\Omega$ and $x_1 = |\vec{x}_1|$ in this case :

Ergo

$$\vec{x}_1 \cdot \vec{x}_2 = |\vec{x}_1| |\vec{x}_2| \cos\Omega$$



$$\vec{T} = \frac{1}{\sqrt{1+\left(\frac{df}{dx}\right)^2}} \left(1, \frac{df}{dx}\right)$$ is tg to the curve.

$\vec{N} = (a,b)$ / $a^2+b^2 = 1$

$\vec{T} \cdot \vec{N} = 0$:

Ergo :

$$\vec{N} = \frac{1}{\sqrt{1+\left(\frac{df}{dx}\right)^2}} \left(\frac{df}{dx}, -1\right)$$

1.3)

Recalling that

$$\vec{T}_1 = \frac{1}{\sqrt{1+(\partial_x f)^2}} (1, 0, \partial_x f)$$

$$\vec{T}_2 = \frac{1}{\sqrt{1+(\partial_y f)^2}} (0, 1, \partial_y f)$$

we search $\vec{N}$ such that: $\vec{N} = (a, b, c)$

$$\begin{cases} |\vec{N}|^2 = 1 \\ \vec{N} \cdot \vec{T}_1 = 0 \\ \vec{N} \cdot \vec{T}_2 = 0 \end{cases} \qquad \begin{cases} a^2 + b^2 + c^2 = 1 \\ a + c(\partial_x f) = 0 \\ b + c(\partial_y f) = 0 \end{cases}$$

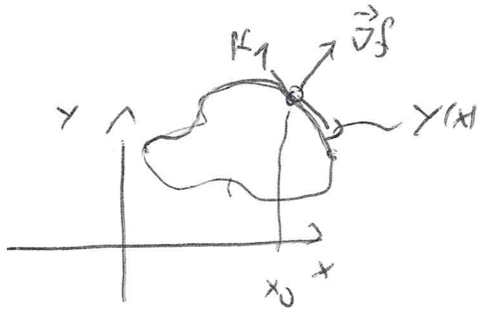
Ergo we find:

$$\vec{N} = \frac{1}{\sqrt{1+(\partial_x f)^2 + (\partial_y f)^2}} \left(-\partial_x f, -\partial_y f, 1 \right)$$

1.4)

$$f(x, y(x)) = z_0 = \text{const.}$$

$$(\vec{\nabla} f)_{\vec{x}_0} = ((\partial_x f)_{\vec{x}_0}, (\partial_y f)_{\vec{x}_0}).$$



Now, if we consider that

$$\vec{T} = \frac{1}{\sqrt{1+y'(x_0)^2}}(1, y'(x_0))$$

we have to show that $\vec{\nabla} f \perp \vec{T}$, that is:

$$\vec{T} \cdot \vec{\nabla} f = 0$$

Ergo:

$$(\partial_x f)_{\vec{x}_0} + (\partial_y f)_{\vec{x}_0} \cdot y'(x_0) = 0$$

This is however easily verified for each point on $K_1 \subset K$

In fact: (K_1 = part of K described by $y(x)$)

$$F(x) = f(x, y(x)) = z_0 = \text{const} \quad \forall (x,y) \in K_1 \subset K$$

$$F'(x) = 0 = (\partial_x f) + (\partial_y f) \cdot y'(x) \quad \forall (x,y) \in K_1$$

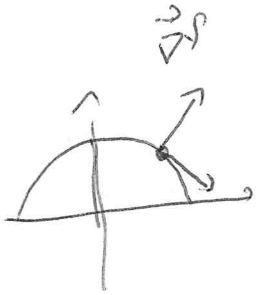
that is also for $(x, y) = (x_0, y_0)$.

1.5) $f(x,y) = x^2+y^2$

$$K = \{ (x,y) \,/\, x^2+y^2 = 2 \}$$

$$y = \pm\sqrt{2-x^2}$$

$(1,1)$ belongs to $y(x) = \sqrt{2-x^2}$



$$f(x, y(x)) = 2 \,!$$

$$\left(\vec{T}(x)\right)_{\vec{x}_0} = \frac{1}{\sqrt{1+y'(1)^2}} \left(1, \frac{-2x}{2\sqrt{2-x^2}}\right)_{\substack{x=1 \\ y=1}} \qquad \frac{1}{\sqrt{2}}(1,-1)$$

$$\vec\nabla f = (2x, 2y)$$

$\left(\vec\nabla f\right)_{\vec{x}_0} = 2(1,1)$ is $\perp$ to $\vec{T}$

$\vec\nabla f \cdot \vec{T} = 2(1,1) \cdot \frac{1}{\sqrt{2}}(1,-1) = 0$ q.e.d.

EX C)

$f(x,y) = xy - x + 1$ $\qquad x(t) = \{2\cos t, \sin t\}$

$F(t) = f(2\cos t, \sin t) = 2\cos t \sin t - 2\cos t + 1$

$F'(t) = 0$

$\hookrightarrow \boxed{\begin{array}{ll} t_0 = 2\pi - \frac{\pi}{6} & \text{MINIMUM} \\ t_0 = \pi + \frac{\pi}{6} & \text{MAXIMUM.} \end{array}}$

In fact:

$F'(t) = -2\sin^2 t + 2\cos^2 t + 2\sin t =$

$= -2\sin^2 t + 2(1 - \sin^2 t) + 2\sin t =$

$= -4 \cdot \sin^2 t + 2\sin t + 2 = 0$

$q = \sin t$

$-4q^2 + 2q + 2 = 0$ $\qquad q_{1,2} = \frac{+1 \pm \sqrt{1 - 4(-2)}}{2 \cdot (-2)}$

$-2q^2 + q + 1 = 0$

$2q^2 - q - 1 = 0$

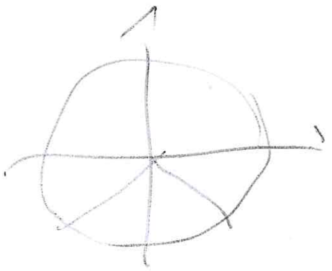
You get:

$q_1 = 1$

$q_2 = -\frac{1}{2}$

$\sin t = 1 \mapsto t = \frac{\pi}{2} \Rightarrow$ But this is a flex

$\sin t = -\frac{1}{2} \to$



$t = 2\pi - \frac{\pi}{6} \to$ MIN

$t = \pi + \frac{\pi}{6} \mapsto$ Max

(Putting it the 2° derivatives you can find that $\pi + \frac{\pi}{6}$ is MAX, $2\pi - \frac{\pi}{6}$ MIN and the $\frac{\pi}{2}$ a FLEX)

$$\int_0^1 dx \int_0^1 dy \, (x^2+y^2-1)^2$$

$$= \int_{-1}^1 dx \int_0^1 dy \, \left(y^4 + 2y^2(x^2-1) + (x^2-1)^2\right)$$

$$= \int_{-1}^1 dx \left[\frac{y^5}{5} + \frac{2y^3}{3}(x^2-1) + y(x^2-1)^2\right]_0^1$$

$$= \int_{-1}^1 dx \left[\frac{8}{15} - \frac{4x^2}{3} + x^4\right] =$$

$$= \left[\frac{8}{15}x - \frac{4}{3}\frac{x^3}{3} + \frac{x^5}{5}\right]_{-1}^1 = \frac{26}{45}$$

$$\int_C dx\,dy\,(x^2+y^2-1)^2 =$$

$$= \int_0^{2\pi} d\varphi \int_0^{1} dr\, r\,(r^2-1)^2 =$$

$$= 2\pi \int_0^1 dr\, r\left(r^4+1-2r^2\right) = 2\pi \int_0^1 dr\,\left(r^5-2r^3+1\right)$$

$$= 2\pi\left[\frac{r^6}{6} - \frac{2r^4}{4} + r\right]_0^1 = 2\pi\left[\frac{1}{6} - \frac{2}{4} + 1\right]$$

$$= 2\pi\left[\frac{1}{6} - \frac{1}{2} + 1\right] = 2\pi\left[\frac{1}{6} + \frac{1}{2}\right] = 2\pi\,\frac{1+3}{6} = \frac{4}{3}\pi$$