System of linear equations

$$\begin{cases} \alpha x + \beta y = x_0 \\ \gamma x + \delta y = y_0 \end{cases}$$

How to solve it? Well, at this level we can do it by substitution.

From the 2°:

$$\delta y = y_0 - \gamma x$$

$$y = \frac{y_0}{\delta} - \frac{\gamma}{\delta} x$$

Plug it into the 1°:

$$\alpha x + \beta \left(\frac{y_0}{\delta} - \frac{\gamma}{\delta} x \right) = x_0$$

$$x \left(\alpha - \frac{\beta \gamma}{\delta} \right) = x_0 - \frac{\beta y_0}{\delta} \Rightarrow x = \frac{x_0 - \frac{\beta y_0}{\delta}}{\alpha - \frac{\beta \gamma}{\delta}}$$

Multiply by δ above and below:

$$\boxed{x = \frac{\delta x_0 - \beta y_0}{\alpha \delta - \beta \gamma}}$$

Plug into the expr. of y:

$$y = \frac{y_0}{\delta} - \frac{\gamma}{\delta}\left(\frac{\delta x_0 - \beta y_0}{\alpha\delta - \beta\gamma}\right) = \frac{y_0(\alpha\delta - \beta\gamma) - \gamma(\delta x_0 - \beta y_0)}{\delta(\alpha\delta - \beta\gamma)}$$

$$= \frac{y_0\alpha\delta - \cancel{y_0\beta\gamma} - \gamma\delta x_0 + \cancel{\gamma\beta y_0}}{\delta(\alpha\delta - \beta\gamma)}$$

$$\boxed{y = \frac{y_0\alpha - \gamma x_0}{\alpha\delta - \beta\gamma}}$$

Note, these solutions make sense only if

$\alpha\delta - \beta\gamma \neq 0$!!!

Seen already this expr?

We can instead rewrite the system in a matrix form

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$A\vec{x} = \vec{x}_0 \qquad A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Now, multiply by the matrix A^{-1} from the left.

$$\underbrace{A^{-1}A}_{\mathbb{1}_2}\vec{x} = A^{-1}\vec{x}_0$$

$$\boxed{\vec{x} = A^{-1}\vec{x}_0}$$

A^{-1} exists only if $\det A \neq 0$... but $\det A = \alpha\delta - \beta\gamma$!!!

$$\vec{x} = A^{-1}\vec{x}_0$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\det A}\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{cases} x = \dfrac{\delta x_0 - \beta y_0}{\det A} \\[2ex] y = \dfrac{-\gamma x_0 + \alpha y_0}{\det A} \end{cases}$$

Indeed this reasoning does not depend on the nr of dimension.

$$A\vec{x} = \vec{x}_0 \quad \text{with } \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}$$

can be solved as

$$\boxed{\vec{x} = A^{-1}\vec{x}_0}$$!!!

Now, it is not always that easy to write down A^{-1} ... already in the 3×3 case the explicit expressions are rather complicated.

Before discussing a trick to solve it, let us discuss two simple cases:

a) $\vec{x}_0 = \vec{0}$.

If $\det A \neq 0 \Rightarrow \exists A^{-1}$ ... then $\vec{x} = A^{-1}\vec{x}_0 = \vec{0}$!

The only solution is $\vec{x} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, i.e. the trivial solution !!!!

a $\det A = 0$

$$\begin{pmatrix} 3 & 4 \\ 6 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$\det A = 8 \cdot 3 - 6 \cdot 4 = 0$!!

The solutions do not make sense... Indeed if we look at it closer we ha

$$3x + 4y = x_0$$

$$6x + 8y = y_0$$

Multiply the first eq. by 2:

$$\begin{cases} 6x + 8y = 2x_0 \\ 6x + 8y = y_0 \end{cases}$$

There are now two possibilities:

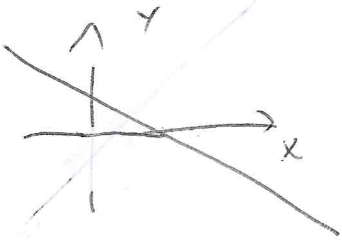
$2x_0 \neq y_0 \Rightarrow$ No sols. at all !!!

$2x_0 = y_0 \Rightarrow$ The two eqs are not indep.

We have as a solution a full line:

$6x + 8y = 2x_0 = y_0 \Rightarrow$ $8y = y_0 - 6x$

$y = \frac{1}{8} y_0 - 6x$



Kramer's rule

$$A\vec{x} = \vec{x}_0$$

$$x_1 = \frac{\det A_1}{\det A}$$

whereas:

$$A_1 = \left(\begin{pmatrix} \vec{x}_0 \end{pmatrix} \overbrace{\begin{pmatrix} \tilde{A} \end{pmatrix}}^{\text{from 2nd up to the Nth column of } A} \right) \qquad \tilde{A} = \begin{pmatrix} A_2^c & A_3^c & \dots & A_N^c \end{pmatrix}$$

The same for all the others variables:

$$x_i = \frac{\det A_i}{\det A}$$

whereas $A_i = \left(A_c^1 \; A_c^2 \dots A_c^{i-1} \begin{pmatrix} \vec{x}_0 \end{pmatrix} A_c^{i+1} \dots A_c^N \right)$

Let us show this explicitely on a 2x2 matrix

$$A\vec{x} = \vec{x}_0$$

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \det A = \alpha\delta - \beta\gamma$$

$$A_1 = \begin{pmatrix} x_0 & \beta \\ y_0 & \delta \end{pmatrix} \qquad \det A_1 = \delta x_0 - \beta y_0$$

$$A_2 = \begin{pmatrix} \alpha & x_0 \\ \gamma & y_0 \end{pmatrix} \qquad \det A_2 = \alpha y_0 - \gamma x_0$$

$$\boxed{\begin{aligned} x &= \frac{\det A_1}{\det A} = \frac{\delta x_0 - \beta y_0}{\det A} \\ y &= \frac{\det A_2}{\det A} = \frac{\alpha y_0 - \gamma x_0}{\det A} \end{aligned}}$$

q.e.d.