Tutorial “General Relativity”

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Sheet No. 4 – Solutions

will be discussed on Dec/13/16

1. Ricci Theorem

The affine connections (Christoffel symbols) are given as

$$\Gamma^{\lambda}{}_{\nu\mu} = \Gamma^{\lambda}{}_{\nu\mu} = \frac{1}{2} g^{\lambda\kappa} \left(g_{\nu\kappa|\mu} + g_{\kappa\mu|\nu} - g_{\mu\nu|\kappa} \right).$$

(a) Show through direct calculation that

$$g_{\mu\nu\|\kappa} = 0, \quad g^{\mu\nu}{}_{\|\kappa} = 0, \quad g^{\mu}{}_{\nu\|\kappa} = 0.$$

Solution: By definition of the covariant derivative we have

$$\begin{aligned} g_{\mu\nu\|\kappa} &= \partial_\kappa g_{\mu\nu} - \Gamma^{\lambda}{}_{\mu\kappa} g_{\lambda\nu} - \Gamma^{\lambda}{}_{\nu\kappa} g_{\mu\lambda} \\ &= \partial_\kappa g_{\mu\nu} - \frac{1}{2} \left(\partial_\mu g_{\nu\kappa} + \partial_\kappa g_{\mu\nu} - \partial_\nu g_{\mu\kappa} \right) - \frac{1}{2} \left(\partial_\nu g_{\mu\kappa} + \partial_\kappa g_{\mu\nu} - \partial_\mu g_{\nu\kappa} \right) \\ &= 0, \end{aligned} \tag{1}$$

because the equally colored terms cancel each other. In the 2nd step we have used

$$\Gamma^{\lambda}{}_{\mu\kappa} g_{\lambda\nu} = g_{\nu\kappa|\mu} + g_{\mu\nu|\kappa} - g_{\mu\kappa|\nu}. \tag{2}$$

For the contravariant components we have

$$\begin{aligned} g^{\mu\nu}{}_{\|\kappa} &= \nabla_\kappa g^{\mu\nu} \\ &= \partial_\kappa g^{\mu\nu} + \Gamma^{\mu}{}_{\kappa\lambda} g^{\lambda\nu} + \Gamma^{\nu}{}_{\kappa\lambda} g^{\mu\lambda} \\ &= \partial_\kappa g^{\mu\nu} + \frac{1}{2} g^{\mu\alpha} g^{\nu\lambda} \left(\partial_\kappa g_{\alpha\lambda} + \partial_\lambda g_{\alpha\kappa} - \partial_\alpha g_{\kappa\lambda} \right) + \frac{1}{2} g^{\nu\alpha} g^{\mu\lambda} \left(\partial_\kappa g_{\alpha\lambda} + \partial_\lambda g_{\alpha\kappa} - \partial_\alpha g_{\kappa\lambda} \right) \\ &= \partial_\kappa g^{\mu\nu} + \frac{1}{2} (g^{\mu\alpha} g^{\nu\lambda} + g^{\nu\alpha} g^{\mu\lambda}) \left(\partial_\kappa g_{\alpha\lambda} + \partial_\lambda g_{\alpha\kappa} - \partial_\alpha g_{\lambda\kappa} \right) \\ &= \partial_\kappa g^{\mu\nu} + \frac{1}{2} (g^{\mu\alpha} g^{\nu\lambda} + g^{\nu\alpha} g^{\mu\lambda}) \partial_\kappa g_{\alpha\lambda}. \end{aligned} \tag{3}$$

In the last step we have used that the red part is antisymmetric under exchange of the index pair (λ, α) and that it is contracted with the left bracket, which is symmetric in this index pair. To further simplify the remaining expression, we note that

$$g^{\mu\alpha} g_{\alpha\lambda} = \delta^\mu_\lambda = \text{const.} \quad \Rightarrow \quad g^{\mu\alpha} \partial_\kappa g_{\alpha\lambda} + g_{\alpha\lambda} \partial_\kappa g^{\mu\alpha} = 0 \;\Rightarrow\; g^{\mu\alpha} \partial_\kappa g_{\alpha\lambda} = -g_{\alpha\lambda} \partial_\kappa g^{\mu\alpha}. \tag{4}$$

Using this identity in (3) yields

$$\begin{aligned} g^{\mu\nu}{}_{\|\kappa} &= \partial_\kappa g^{\mu\nu} - \frac{1}{2} \left(g^{\mu\alpha} g_{\alpha\lambda} \partial_\kappa g^{\nu\lambda} + g^{\nu\alpha} g_{\alpha\lambda} \partial_\kappa g^{\mu\lambda} \right) \\ &= \partial_\kappa g^{\mu\nu} - \frac{1}{2} (\delta^\mu_\lambda \partial_\kappa g^{\nu\lambda} + \delta^\nu_\lambda \partial_\kappa g^{\mu\lambda}) = 0. \end{aligned} \tag{5}$$

To prove also the last formula, we use the fact that $g^{\mu}{}_{\nu} = \delta^\mu_\nu = \text{const}$ and thus

$$g^{\mu}{}_{\nu\|\kappa} = \nabla_\kappa \delta^\mu_\nu = \Gamma^{\mu}{}_{\kappa\lambda} \delta^\lambda_\nu - \Gamma^{\lambda}{}_{\kappa\nu} \delta^\mu_\lambda = \Gamma^{\mu}{}_{\kappa\nu} - \Gamma^{\mu}{}_{\kappa\nu} = 0. \tag{6}$$

(b) Show the validity of product rule for the covariant derivative on the example ${T^\mu}_\nu = A^\mu B_\nu$, i.e.,

$$ {T^\mu}_{\nu\|\rho} = {A^\mu}_{\|\rho} B_\nu + A^\mu B_{\nu\|\rho}. $$

Solution: Using the definition of the covariant derivative gives

$$ \begin{aligned} {T^\mu}_{\nu\|\rho} &= \partial_\rho {T^\mu}_\nu + {\Gamma^\mu}_{\rho\sigma} {T^\sigma}_\nu - {\Gamma^\sigma}_{\rho\nu} {T^\mu}_\sigma \\ &= \partial_\rho (A^\mu B_\nu) + {\Gamma^\mu}_{\rho\sigma} A^\sigma B_\nu - {\Gamma^\sigma}_{\rho\nu} A^\mu B_\sigma \\ &= B_\nu \partial_\rho A^\mu + A^\mu \partial_\rho B_\nu + {\Gamma^\mu}_{\rho\sigma} A^\sigma B_\nu - {\Gamma^\sigma}_{\rho\nu} A^\mu B_\sigma \\ &= \left(\partial_\rho A^\mu + {\Gamma^\mu}_{\rho\sigma} A^\sigma\right) B_\nu + A^\mu \left(\partial_\rho B_\nu - {\Gamma^\sigma}_{\rho\nu} A^\mu B_\sigma\right) \\ &= {A^\mu}_{\|\rho} B_\nu + A^\mu B_{\nu\|\rho}, \end{aligned} \tag{7} $$

and this what we wanted to show.

(c) Why can one "naively" lower and raise indices in covariant derivatives, i.e., why is for, e.g., a tensor $T_{\mu\nu}$

$$ {T^\mu}_{\nu\|\rho} = g^{\mu\sigma} T_{\sigma\nu\|\rho}. $$

Solution: Using the result of part (a) and the validity of the product rule for covariant derivatives, proven in part (b), we immediately get

$$ {T^\mu}_{\nu\|\rho} = (g^{\mu\sigma} T_{\sigma\nu})_{\|\rho} \stackrel{\text{(b)}}{=} {g^{\mu\sigma}}_{\|\rho} T_{\sigma\nu} + g^{\mu\sigma} T_{\sigma\nu\|\rho} \stackrel{\text{(a)}}{=} + g^{\mu\sigma} T_{\sigma\nu\|\rho}. \tag{8} $$

(d) Show that for the "covariant curl" for any vector field A_μ one can use the partial derivatives instead of the covariant ones:

$$ F_{\mu\nu} = A_{\nu\|\mu} - A_{\mu\|\nu} = A_{\nu|\mu} - A_{\mu|\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. $$

Solution: Using the covariant derivative,

$$ A_{\nu\|\mu} = \nabla_\mu A_\nu = \partial_\mu A_\nu - {\Gamma^\rho}_{\mu\nu} A_\rho \tag{9} $$

shows that

$$ F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{10} $$

because of ${\Gamma^\rho}_{\mu\nu} = {\Gamma^\rho}_{\nu\mu}$, i.e., because the pseudo-Riemannian spacetime manifold of general relativity is torsion free.

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2. Ideal fluid

The non-relativistic hydrodynamical equations describing mass-, and- momentum energy conservation, for an ideal fluid are given by

$$ \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0, \tag{11} $$

$$ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} + \frac{\vec{\nabla} P}{\rho} = 0, \tag{12} $$

$$ \frac{\partial \epsilon}{\partial t} + \vec{\nabla} \cdot (\epsilon \vec{v}) + P \vec{\nabla} \cdot \vec{v} = 0. \tag{13} $$

(a) Express the energy density ϵ and the Pressure P of the ideal fluid as a function of the mass density density ρ and temperature T.

Solution: We use the equations of state of an ideal gas,

$$ PV = k_{\mathrm{B}} N T, \quad U = \frac{f}{2} N k_{\mathrm{B}} T, \tag{14} $$

where $f = 3$ for a gas of monatomic, $f = 5$ of diatomic molecules, and $f = 6$ for a gas with molecules consisting of at least three atoms. We assume that we are at temperatures such that vibrational degrees of freedom are not active. Dividing the equations by V and using $\rho = mN/V$, where m is the mass of one gas molecule we get

$$P = \frac{k_{\mathrm{B}}\rho T}{m}, \quad \epsilon = \frac{U}{V} = \frac{f}{2}\frac{k_{\mathrm{B}}\rho T}{m}. \tag{15}$$

(b) Show via Eq. (13) that an isothermal ideal fluid, i.e., a fluid for which $T(t, \vec{x}) = T_0$, is also incompressible, meaning $\vec{\nabla} \cdot \vec{v} = 0$.

Solution: Since $T = \text{const}$, plugging in (15) in (13)

$$\frac{f k_{\mathrm{B}} T}{2m}[\partial_t \rho + \vec{\nabla}(\rho \vec{v})] + P\vec{\nabla} \cdot \vec{v} = 0. \tag{16}$$

Due to (11) the square bracket vanishes, and since $P > 0$ this shows that indeed $\vec{\nabla} \cdot \vec{v} = 0$.

Note: The condition $\operatorname{div} \vec{v} = \vec{\nabla} \cdot \vec{v} = 0$ means the incompressibility of the fluid. An incompressible fluid is defined by the condition that the density $\rho = \text{const}$, i.e., independent of time and position. This implies $\partial_t \rho = 0$ and $\vec{\nabla}\rho = 0$. From the continuity eqution (11) this implies

$$0 = \partial_t \rho + \vec{\nabla}(\rho \vec{v}) = \vec{\nabla}(\rho \vec{v}) = \partial_j(\rho v_j) = v_j \partial_j \rho + \rho \partial_j v_j = \rho \partial_j v_j = \rho \vec{\nabla} \cdot \vec{v} \;\Rightarrow\; \vec{\nabla} \cdot \vec{v} = 0. \tag{17}$$
