Tutorial "General Relativity"

Winter term 2016/2017

Lecturer: Prof. Dr. C. Greiner
Tutor: Hendrik van Hees

Sheet No. 7 – Solutions

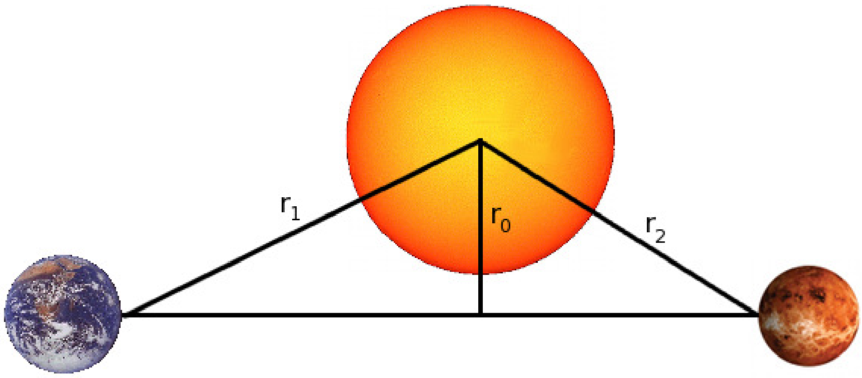
will be discussed on February 07, 2017

1. Shapiro Delay

Consider a light beam that moves from Earth to Venus and back, coming close to the Sun (see the figure). Non-relativistically the total "radar time" needed to come back to Earth obviously is

$$t_{\text{nrel}} = 2\left(\sqrt{r_1^2 - r_0^2} + \sqrt{r_2^2 - r_0^2}\right), \tag{1}$$

where r_0 is the distance of closest approach of the light beam to the Sun. The task of this problem is to evaluate the radar time taking into account the gravity of the Sun according to General Relativity.



Start from the Schwarzschild invariant-length element,

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right) c^2 \mathrm{d}t^2 - \frac{\mathrm{d}r^2}{(1 - \frac{2m}{r})} - r^2(\mathrm{d}\phi^2 + \sin^2\theta \mathrm{d}\phi^2), \quad 2m = r_S = \frac{2GM}{c^2}, \tag{2}$$

and use the result for the null geodesic from the lecture, describing the light beam (by an appropriate choice of the coordinate system according to the initial conditions running in the plane $\vartheta = \pi/2 = \text{const}$),

$$\left(1 - \frac{2m}{r}\right) c^2 \dot{t}^2 - \frac{\dot{r}^2}{(1 - \frac{2m}{r})} - \frac{h^2}{r^2} = 0. \tag{3}$$

The dot indicates the derivative with respect to an arbitrary affine parameter[1]. Here

$$h = r^2 \dot{\phi} = \text{const} \tag{4}$$

follows from the rotational symmetry formalized by the fact that ϕ is a cyclic variable of the Lagrangian, $L = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu/2$ leading to the equation of the geodesic. The same holds true for t since the Schwarzschild metric is static, i.e.,

$$\left(1 - \frac{2m}{r}\right) \dot{t} = A = \text{const}. \tag{5}$$

[1]Note that for a light beam, which formally follows the geodesic of a massless particle, there does not exist a physical definition of such a parameter, like proper time for a massive particle!

(a) Express h in terms of r_0 and A, evaluate $\mathrm{d}r/\mathrm{d}t = \dot{r}/\dot{t}$ from (3) and (4).

Solution: Since r_0 is the radius of closest approach of the geodesic to the center at this point $\dot{r} = 0$. Using this in (3) one finds, using (5)

$$\frac{A^2c^2}{1-2m/r_0} - \frac{h^2}{r_0^2} = 0 \Rightarrow \frac{h^2}{A^2} = \frac{r_0^2 c^2}{1-2m/r_0}. \tag{6}$$

(b) With that result show, that the radar time for the signal running forth and back from Venus is given by[2]

$$ct_{\text{radar}} = 2[F(r_1) + F(r_2)], \tag{7}$$

where

$$F(r_1) = \int_{r_0}^{r_1} \frac{\mathrm{d}r}{1-\frac{2m}{r}} \left[1 - \frac{1-2m/r}{1-2m/r_0}\left(\frac{r_0}{r}\right)^2\right]^{-1/2}, \quad r_1 > r_0. \tag{8}$$

Solution: Dividing (3) by $\dot{t}^2$ and a little algebra leads to

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \left(\frac{\dot{r}}{\dot{t}}\right)^2 = \left(1-\frac{2m}{r}\right)^2 - \left(1-\frac{2m}{r}\right)\frac{r^2\dot{\phi}^2}{c^2\dot{t}^2}. \tag{9}$$

For the last term we have with (4) and (5)

$$\frac{r^2\dot{\phi}^2}{\dot{t}^2} = \frac{h^2}{r^2}\left(1-\frac{2m}{r}\right)^2 \frac{1}{A^2c^2} = \left(1-\frac{2m}{r}\right)^2 \frac{r_0^2}{r^2(1-2m/r_0)}, \tag{10}$$

and inserting this into (9) leads to

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \left(\frac{\dot{r}}{\dot{t}}\right)^2 = c^2\left(1-\frac{2m}{r}\right)^2 \left[1 - \left(\frac{r_0}{r}\right)^2 \frac{1-2m/r}{1-2m/r_0}\right]. \tag{11}$$

Taking the inverse of this equation then leads to (8). Concerning the sign of the square root we have to choose it such that $\mathrm{d}t > 0$ along the entire geodesic, i.e., the integral must be positive, as is the case for (8).

(c) Evaluate the integral (8) approximately by expanding up to linear order in m, assuming that $2m/r = r_S/r \ll 1$ along the entire worldline of the light beam.

Hint: The result is

$$F(r_1) = \sqrt{r_1^2 - r_0^2} + 2m\ln\left(\frac{r_1 + \sqrt{r_1^2 - r_0^2}}{r_0}\right) + m\sqrt{\frac{r_1 - r_0}{r_1 + r_0}} + \mathcal{O}(m^2). \tag{12}$$

Solution: Taylor expansion of the integrand in (8) with respect to m gives

$$\frac{1}{1-\frac{2m}{r}}\left[1 - \frac{1-2m/r}{1-2m/r_0}\left(\frac{r_0}{r}\right)^2\right]^{-1/2} = \frac{r}{\sqrt{r^2 - r_0^2}}\left[1 + \frac{m(2r+3r_0)}{r(r+r_0)}\right] + \mathcal{O}(m^2). \tag{13}$$

Evaluating the integral gives (12). Finally (7) leads to the formula for the Shapiro delay, i.e., the additional time needed by the radar signal to reach the Earth again being reflected from Venus is given by

$$\begin{aligned} c\Delta t_{\text{Shapiro}} &= c(t_{\text{radar}} - t_{\text{nrel}}) \\ &= 4m\ln\left[\frac{(r_1 + \sqrt{r_1^2 - r_0^2})(r_2 + \sqrt{r_2^2 - r_0^2})}{r_0^2}\right] + 2m\left(\sqrt{\frac{r_1 - r_0}{r_1 + r_0}} + \sqrt{\frac{r_1 - r_0}{r_1 + r_0}}\right). \end{aligned} \tag{14}$$

[2] We neglect the motion of the planets during the very short travel time of the light beam.

Note: For completeness we give a complete derivation of the above used results for the null geodesics in Schwarzschild spacetime. The Lagrangian for the geodesics in affine parametrization reads

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} \left[\left(1 - \frac{2m}{r}\right) c^2 \dot{t}^2 - \frac{\dot{r}^2}{1 - 2m/r} - r^2 \dot{\phi}^2 \right]. \tag{15}$$

Here we have already used that due to rotational invariance the motion is always in a plane, and we choose the equator plane of the spherical coordinates, $\vartheta = \pi/2$, as this plane. It's obvious that the coordinates t and ϕ are "cyclic", i.e., their canonical momenta are conserved, leading to

$$p_t = \frac{\partial L}{\partial \dot{t}} = \left(1 - \frac{2m}{r}\right) c^2 \dot{t} = A c^2 = \text{const.}, \tag{16}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = -r^2 \dot{\phi} = -h = \text{const.} \tag{17}$$

Finally, since L does not explicitly depend on the world-line parameter λ and since L is quadratic in the $\dot{x}^\mu$ it is a conserved quantity, and thus λ an affine parameter of the trajectory (which is a geodesic in Schwarzschild spacetime).

Since we want to investigate the "null geodesics", i.e., the trajectory of a freely falling massless particle, we have $L = 0$:

$$\left(1 - \frac{2m}{r}\right) c^2 \dot{t}^2 - \frac{\dot{r}^2}{1 - 2m/r} - r^2 \dot{\phi}^2 = 0. \tag{18}$$

First we use (17) to eliminate $\dot{\phi}$, which leads to (3), which was the starting point for our calculation above:

$$\left(1 - \frac{2m}{r}\right) c^2 \dot{t}^2 - \frac{\dot{r}^2}{\left(1 - \frac{2m}{r}\right)} - \frac{h^2}{r^2} = 0. \tag{19}$$