Tutorial "General Relativity"

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Sheet No. 5

will be discussed on Jan/17/17

1. Barometric formula in General Relativity

In the lecture we have shown that an ideal fluid obeys the equation of motion,

$$\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\beta}\left[\sqrt{|g|}\left(\varrho+\frac{p}{c^2}\right)u^\alpha u^\beta\right]+\left(\varrho+\frac{p}{c^2}\right)\Gamma^\alpha_{\beta\nu}u^\beta u^\nu-g^{\alpha\beta}\frac{\partial p}{\partial x^\beta}=0. \tag{1}$$

Now consider an ideal fluid in a stationary gravitational field in hydrostatic ($u^\alpha=0$ for $\alpha\in\{1,2,3\}$) equilibrium. This case is relevant for the description of neutron stars, which have a strong gravitational field and thus have to be treated within the General Theory of Relativity.

(a) Show that $u^0=\frac{c}{\sqrt{g_{00}}}$.

(b) Explain, why the first term in (1) is identically zero.

(c) Prove that the equation simplifies to a generalized barometric formula,

$$\frac{\partial p}{\partial x^\beta}=-\frac{1}{2}\left(\varrho c^2+p\right)\frac{\partial g_{00}}{\partial x^\beta}\frac{1}{g_{00}}=0, \tag{2}$$

i.e., for the spatial components,

$$\partial_j p=-\frac{\varrho c^2+p}{2}\partial_j(\ln g_{00})\quad\text{for}\quad j\in\{1,2,3\}. \tag{3}$$

(d) Show that in the non-relativistic limit

$$\vec{\nabla}p=-\varrho\vec{\nabla}\phi_{\text{grav}}, \tag{4}$$

where ϕ_{grav} denotes the Newtonian gravitational potential.

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2. Geodesics equation from ideal hydrodynamics

In this problem we want the show that the equations of motion for free fall leads to the equation for (timelike) geodesics via hydrodynamics of an ideal fluid. The generally covariant equations of motion of an ideal fluid is given by the local conservation of energy and momentum, i.e.,

$$T^{\mu\nu}{}_{\|\nu}=0 \tag{5}$$

for the energy-momentum tensor

$$T^{\mu\nu}=(\epsilon+P)u^\mu u^\nu-Pg^{\mu\nu}, \tag{6}$$

with the energy density ϵ, pressure P, and four-velocity field u^μ normalized such that $u_\mu u^\mu=1$.

(a) Show that

$$u^\mu{}_{\|\nu}u_\mu=0. \tag{7}$$

(b) Show from (5) and the contraction of the equation with u_μ that

$$(\epsilon + P) {u^\mu}_{\|\nu} u^\nu = (g^{\mu\nu} - u^\mu u^\nu) P_{\|\nu}. \tag{8}$$

(c) For "dust", i.e., a fluid consisting of a non-interacting particles, which means that $P = 0$, this obviously implies that

$$ {u^\mu}_{\|\nu} u^\nu = 0. \tag{9}$$

Show that this implies that the flow lines, i.e., the trajectories of the dust particles, are geodesics.

Hint: The flow-lines are defined by the differential equation

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}s} = u^\mu. \tag{10}$$

Note: The shown results show that energy-momentum conservation for freely falling particles implies that they follow (timelike) geodesics of spacetime, as is also implied by the (weak) equivalence principle used in the lecture to motivate the description of gravity as the curvature of a non-Euclidean spacetime manifold.

According to (8) the flow lines of a free falling ideal fluid are no geodesics, but on the fluid particles an additional force is acting due to its pressure.