Electrical Conductivity of QGP in Strong Magnetic Field

Daisuke Satow
Collaborator: Koichi Hattori (Fudan Uni, Shanghai 🇨🇳)
Shiyong Li, Ho-Ung Yee (Uni. Illinois at Chicago 🇺🇸)

Outline

• (Long) Introduction
  —quarks and gluons in strong B

• Calculation of Conductivity and Results

• Possible Phenomenological Implications
  (Very Brief, more like future works)

• Summary and future perspective
Introduction

Strong magnetic field ($B$) may be generated in heavy ion collision due to Ampere’s law.
Introduction

strength:

$$\sqrt{eB} \sim 100 \text{ [MeV]}$$

(Comparable with $T$)

At thermalization time ($\sim 0.5 \text{ fm}$), there still may be strong $B$.

Heavy ion collision may give a chance to investigate QCD matter at finite temperature in strong magnetic field.
How the system behaves when the energy scale of $B$ is much larger than the typical energy scale of the system?

$(\sqrt{eB} \gg T, m, \Lambda_{\text{QCD}} \ldots)$
Quark in Strong $B$

One-particle state of quark in magnetic field

**Classical: Cyclotron motion due to Lorentz force**
Quark in Strong B

Quantum: Landau Quantization

Longitudinal: Plane wave
Transverse: Gaussian

\[ E_n = \sqrt{(p_z)^2 + m^2 + 2eB \left( n + \frac{1}{2} \right)} \]

The gap (~\(\sqrt{eB}\)) is generated by zero-point oscillation.
For spin-1/2 particle, we have Zeeman effect:

\[ E_n = (p_z)^2 + m^2 + 2eB \left(n + \frac{1}{2} \pm \frac{1}{2}\right) \]

When \( n=0 \) (LLL), gap is small (\( m \sim 1\text{MeV} \)). When \( n>0 \), gap is large (\( \sim \sqrt{eB} \sim 100\text{MeV} \))
Lowest Landau Level (LLL) Approximation

When the typical energy of particle ($T$) is much smaller than gap ($\sqrt{eB}$), the higher LL does not contribute ($\sim \exp(-\sqrt{eB}/T)$), so we can focus on the LLL.

Confined in one direction fermion, no spin degrees of freedom.

$$E_n = \sqrt{(p_z)^2 + m^2}$$

In heavy-ion collision, this condition may be marginally realized ($T \sim \sqrt{eB} \sim 100\text{MeV}$). But in Weyl semi-metal, it is already realized ($T \sim 1\text{meV}, \sqrt{eB} \sim 10\text{eV}$).
Gluon in Strong B

Gluon does not have charge, so it does not feel $B$ in the zeroth approximation.

Massless boson in (3+1)D
Gluon in Strong B

Coupling with (1+1)D quarks generates gluon mass. (Schwinger mass generation)

\[ M^2 \equiv \frac{1}{2} \times \left( \frac{g^2}{\pi} \sum_{\ell} \frac{|eB|}{2\pi} \right) \sim g^2 eB \]

Color factor

(surface density)

\sim (average distance)^{-2} \sim eB

Schwinger mass

Landau degeneracy

Motivation to Discuss Electrical Conductivity

Electrical conductivity is phenomenologically important because

- Input parameter of magnetohydrodynamics (transport coefficient)

- May increase lifetime of $B$ (Lenz’s law)

\[ \nabla \times \mathbf{E} = -\partial_t \mathbf{B} \]

\[ \partial_t \mathbf{E} = \nabla \times \mathbf{B} - \mathbf{j} \]

When $\sigma$ is large
Possible Scattering Process for Conductivity

\( B=0 \)

1 to 2 scattering is kinematically forbidden; one massless particle can not decay to two massless particles

**strong** \( B \)

Gluon is effectively massive in \((1+1)D\)

\[ E = \sqrt{p_z^2 + p_{\perp}^2 + M^2} \]

Decay of a gluon into two quarks becomes kinematically possible.
Chirality in (1+1)D

Spin is always up.

When $m=0$, the direction of $p_z$ determines chirality.
Chirality in (1+1)D

Chirality is conserved at $m=0$:

$\chi = +1 + 1 = 2$

$\chi = 0$

1 to 2 scattering is forbidden at $m=0$. 
Motivation to Discuss Electrical Conductivity

Electrical conductivity is also theoretically interesting: the scattering process is very different from that in $B=0$.

- Because the kinematics is non-standard (1+1 D for quark, 3+1D for gluon), the 1 to 2 scattering is the leading process, instead of 2 to 2.

- At $m=0$, the 1 to 2 process is forbidden due to chirality conservation. Thus, we need to include finite $m$ effect to have non-divergent conductivity.
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Hierarchy of Energy Scale at LLL

- $\sqrt{eB}$: Energy gap of higher LL
- $T$: Typical energy of quark and gluon
- $m$: Chirality flipping amplitude
Hierarchy of Energy Scale at LLL

For ordering of $m$ and $M$, we consider the both cases. 

$$ (m \ll M \text{ and } m \gg M) \quad \quad (M \sim g \sqrt{eB}) $$
Electrical Conductivity

**Conductivity at weak $B$ ($\sqrt{eB} \ll T$)**

$B=0$

$$j^i = \sigma^{ij} E^j$$

$$j_i = \sigma_0 E$$

$\sigma^{ij} = \begin{bmatrix} \sigma_0 & 0 & 0 \\ 0 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix}$

$\sigma_0$ is independent from $B$. ($\sigma_0 \sim e^2 T / g^4$)

$\sigma_1$ is linear in $B$. ($\sigma_1 \sim e^3 B \mu / g^8 T^2$)

**Linear in $B$**

$$j = \sigma_1 E \times B$$

$\sigma^{ij} = \begin{bmatrix} \sigma_0 & \sigma_1 & 0 \\ -\sigma_1 & \sigma_0 & 0 \\ 0 & 0 & \sigma_0 \end{bmatrix}$

Electrical Conductivity

**Strong $B$ (LLL)**

Quarks are confined in the direction of $B$, so there is no current in other directions.

$\sigma^{33}$ is finite, other components are zero. (Very different from weak $B$ case)

$$j^i = \sigma^{ij} E^j$$

$$\sigma^{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$
Calculation of conductivity

**Thermal equilibrium in strong $B$**

$n^f(k^3, T, Z) = n_F(\epsilon_k^L)$

$\epsilon_k^L \equiv \sqrt{(k^3)^2 + m^2}$

**Linear response against $E$**

**Slightly non-equilibrium, finite $j$**

$n^f(k^3, T, Z) = n_F(\epsilon_k^L) + \delta n^f(k^3, T, Z)$

\[
j^3(T, Z) = 2e \sum_f q_f N_c \frac{|e q_f B|}{2\pi} \int \frac{dk^3}{2\pi} v^3 \delta n^f(k^3, T, Z) = \sigma^{33} E^3
\]

**Evaluation of $\delta n_F$ is necessary.**
Calculation of conductivity

Evaluate $n_F$ with $(1+1)$D Boltzmann equation

\[
[\partial_T + v^3 \partial_Z + e q_f E^3(T, Z) \partial_{k^3}] n^f(k^3, T, Z) = C[n]
\]

\[
v^3 \equiv \frac{\partial \epsilon_k^L}{\partial k^3} = \frac{k^3}{\epsilon_k^L}
\]

1 to 2 collision:

\[
C[n] = \frac{1}{2\epsilon_k^L} \int |M|^2 [n_B^{k+l}(1 - n_F^k)(1 - n_F^l) - (1 + n_B^{k+l})n_F^k n_F^l]
\]

\[
|M|^2 = 4g^2 C_f m^2
\]

**Vanishes at $m=0$!**  
(chirality conservation)
Calculation of conductivity

\[ [\partial_T + v^3 \partial_Z + e q_f E^3(T, Z) \partial_{k^3}] n^f(k^3, T, Z) = C[n] \]

linearize \hspace{1cm} n^f(k^3, T, Z) = n_F(\epsilon_k^L) + \delta n^f(k^3, T, Z)

Constant \( E: \partial_T, \partial_Z=0 \)

\[ e q_f E^3 \beta v^3 n_F(\epsilon_k^L)[1 - n_F(\epsilon_k^L)] = C[\delta n^f(k^3, T, Z)] \]

\[ C[n] = \frac{1}{2\epsilon_k^L} \int |M|^2 [n_B^{k+l}(1 - n_F^k)(1 - n_F^l) - (1 + n_B^{k+l})n_F^k n_F^l] \]

linearize

\[ C[\delta n] = -\frac{1}{2\epsilon_k^L} \int |M|^2 [\delta n_F^k (n_B^{k+l} + n_F^l) - \delta n_F^l (n_B^{k+l} + n_F^k)] \]

damping rate of quark \((-2\xi_k \delta n_F^k)\)
Calculation of conductivity

Solution for $\delta n^F$ with damping rate $\xi_k$

$$
\delta n_{F}^{k} = - \frac{1}{2\xi_k} e q_f E^3 \beta v^3 n_F(\epsilon^L_k)[1 - n_F(\epsilon^L_k)]
$$

$$
\dot{j}^3(T, Z) = 2e \sum_f q_f N_c \frac{|B_f|}{2\pi} \int \frac{dk^3}{2\pi} v^3 \delta n_f(k^3, T, Z)
$$

$$
\dot{j}^3 = e^2 \sum_f q_f^2 N_c \frac{|eq_f B|}{2\pi} 4\beta \int \frac{dk^3}{2\pi} (v^3)^2 \frac{1}{2\xi_k} n_F(\epsilon^L_k)[1 - n_F(\epsilon^L_k)] E^3
$$

$$
\sigma^{33}
$$
Quark Damping Rate

\[ \epsilon_k \xi_k = \frac{g^2 C_F m^2}{4\pi} \int_m^\infty dl^0 \frac{n_F(l^0) + n_B(l^0 + \epsilon_k^L)}{\sqrt{(l^0)^2 - m^2}} \]

leading-log approximation \((\ln[T/m]>>1)\)

\[ l^0<<T \text{ dominates} \]

\[ \epsilon_k^L \xi_k \simeq \frac{g^2 C_F m^2}{4\pi} \left[ \frac{1}{2} + n_B(\epsilon_k^L) \right] \int_m^\infty dl^0 \frac{1}{\sqrt{(l^0)^2 - m^2}} \]

\[ \simeq \frac{g^2 C_F m^2}{4\pi} \left[ \frac{1}{2} + n_B(\epsilon_k^L) \right] \ln \left( \frac{T}{m} \right) \]

matrix element

cf: 2 to 2

\[ \sim g^4 \]

soft fermion and hard boson

\[ n_F(1+n_B)+(1-n_F)n_B=n_F+n_B \]
Due to chirality conservation, collision is forbidden when \( m=0 \). Thus, \( \sigma \sim 1/m^2 \).

Despite \( T \gg m \), it is very sensitive to \( m \)!

When \( M \gg m \), \( \ln(T/m) \to \ln(T/M) \).
Other Term Does Not Contribute

\[ C[\delta n] = -\frac{2g^2C_Fm^2}{\epsilon_k^L} \int \left[ \delta n_F^k(n_B^{k+l} + n_F^{l}) - \delta n_F^l(n_B^{k+l} + n_F^{k}) \right] \]

\[ \delta n_F^l = -\frac{eqf}{2\xi_l} E^3 \partial_l n_F(\epsilon_l^L) : \text{odd in } l^3 \]

(Other term)\sim \int (n_B^{k+l} + n_F^{k}) \delta n_F^l \quad \text{even in } l^3 \quad 0

Same for \( m<<M \) case.

Our result (only retaining quark damping rate) is correct.
Equivalent Diagrams

Our calculation is based on (unestablished) (1+1)D kinetic theory, but actually \textit{we can reproduce the same} result by field theory calculation.


Kubo formula: \[ \sigma^{ij} \equiv \lim_{\omega \to 0} \frac{\Pi^{Rij}(\omega)}{i\omega} \]

\[ j^\mu \equiv e \sum_f q_f \bar{\psi}_f \gamma^\mu \psi_f \]
\( f \): flavor index, \( q_r \): electric charge

\[ \Pi^{R\mu\nu}(x) \equiv i\theta(x^0) \langle [j^\mu(x), j^\nu(0)] \rangle \]

\( p \to 0 \) limit
Equivalent Diagrams

**Kinetic eq.**

\[ \partial_X \sim p \]

\[ k= (\epsilon^L_k, \vec{k})= \epsilon^L_k (1, \vec{v}) \]

Connect the ends of lines

\[ (v \cdot \partial_X + 2 \xi_k) \delta n^f = -e q_f E^3 \partial_k n^f (k) \]

**Field Theory**

\[ \frac{1}{(k + p)^2 - m^2 + 2i \xi_{k+p}(k^0 + p^0)} \]

on-shell condition: \( k^2 - m^2 = 2i \xi_k k^0 \), \( p \to 0 \) limit

\[ \frac{1}{2 (k \cdot p + 2i \xi_k k^0)} \]

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Possible Phenomenological Implications

1. Order Estimate

\[ \sigma^{33} = e^2 \sum_f q_f^2 N_c \frac{|e q_f B|}{2\pi} \frac{4T}{g^2 C_F m^2 \ln(T/M)} \]

Because of \( m^{-2} \) dependence, \( s \) contribution is very small.

\[ \alpha_s = \frac{g^2}{4\pi} = 0.3, \]

\[ m = 3\text{MeV}(u, d), 100\text{MeV}(s), \]

\[ eB = 10m^2\pi = (440\text{MeV})^2. \]

\[ M=160\text{MeV} \gg m \]


Beyond LLL approximation, there are also spin down particles, so the scattering is not suppressed by $m^2$.

$\sigma^{33}$ is expected to be smaller at large $T$, so that it smoothly connects with $B=0$ result.
Possible Phenomenological Implications

2. Soft Dilepton Production


\[
\frac{d\Gamma}{d^4p} = \frac{\alpha}{12\pi^4\omega^2} T \sigma^{33}
\]

\(\cdot\) (virtual photon emission rate) \(\sim n_B(\omega)Im\Pi^{\mu\mu} \sim T\sigma^{33}\)

\(\sigma^{33}\) is large

\(e\sqrt{eB} \ll \omega \ll \frac{g^2m^2}{T} \ln\left(\frac{T}{M}\right)\)

\[\text{Soft dilepton production is enhanced by } B?\]
Possible Phenomenological Implications

3. Back Reaction to EM Fields

\[ \partial_t B < 0 \]

**Bad news:**
In LLL approximation, we have no current in transverse plane, so Lenz’s law does NOT work!
The lifetime of $B$ does not increase…
Summary

• We calculated electrical conductivity in strong B using the LLL approximation, for $m \ll M$ and $m \gg M$ cases.

\[ \sigma^{33} = e^2 \sum_f q_f^2 N_c \frac{e q_f B}{2\pi} \frac{4T}{g^2 C_F m^2 \ln(T/m)} \]

Quark density in 1D

Landau degeneracy

Quark damping rate

When $M \gg m$, $\ln(T/m) \rightarrow \ln(T/M)$.

• We found that the conductivity is enhanced by large $B$, and small $m$. The sensitivity to $m$ was explained in terms of chirality conservation.
Future Perspective

• Go beyond LLL approximation (more realistic $B$)

• Calculate other transport coefficients (viscosity, heat conductivity… )
Back Up
Electrical Conductivity

Axial anomaly: \[ \partial_t n_A = \frac{e^2 N_c N_F}{2\pi^2} E \cdot B - \frac{1}{\tau_R} n_A \]

Stationary solution: \[ n_A = \frac{e^2 N_c N_F}{2\pi^2} E \cdot B \tau_R \]

Chiral magnetic effect: \[ J = \frac{e^2 N_c N_F}{2\pi^2} \mu_A B = \frac{e^2 N_c N_F}{2\pi^2 \chi} n_A B \]

\[ \sigma_{zz} = \frac{e^4 (N_c N_F)^2 B^2}{4\pi^4 \chi} \tau_R \]

\[ \chi = N_c \frac{1}{2\pi} \left( \frac{eB}{2\pi} \right) \]

1. Sphaleron

2. Current quark mass

\[ \sigma_{zz} \sim e^2 N_c (eB)T \frac{1}{\alpha_s m_q^2} \]
Collision-dominant case — Boltzmann equation

**Ohmic current**

$E$  

exists even when $\mu=0$.

**Hall current**

$E$  

$cancels when $\mu=0$.  

$j=0$
Motivation

Compute electrical conductivity in strong $B$ limit!
We also assume that the gluon screening mass \((M \sim g\sqrt{eB})\) is much smaller than \(T\), so that the gluon is thermally excited.