Chiral spirals and their fluctuations

1. Standard phase diagram in T & \( \mu \): critical end-point (CEP)

   \textit{Not} seen from lattice at small \( \mu \)

2. Quarkyonic phase at large \( N_c \) (analytic) and \( N_c = 2 \) (lattice)

3. Chiral Spirals in Quarkyonic matter: sigma models, SU(N) and U(1)

4. Phase diagram: \textit{just a 1st order line,}

   with \textit{large} fluctuations in the Lifshitz regime
“Standard” phase diagram for QCD in T & $\mu$: CEP?

Lattice: at quark chemical potential $\mu = 0$, crossover at $T_{ch} \sim 154$ MeV

At $\mu \neq 0$, quarks *might* change scalar 4-pt coupling $< 0$, so transition 1st order

Must meet at a **Critical End Point (CEP)**, *true* 2nd order phase transition

Asakawa & Yazaki ‘89, Stephanov, Rajagopal & Shuryak ‘98 & ‘99
The Phases of QCD

- **Early Universe**
  - Future LHC Experiments

- **Current RHIC Experiments**

- **Crossover**

- **Critical Point**

- **Quark-Gluon Plasma**

- **Hadron Gas**

- **Vacuum**

- **1st order phase transition**

- **Future FAIR Experiments**

- **Color Superconductor**

- **Nuclear Matter**

- **Neutron Stars**

**Temperature**

- ~170 MeV

**Baryon Chemical Potential**

- 0 MeV

- 900 MeV
Lifshitz phase diagram for QCD

Instead: “Lifshitz regime”: strongly coupled, large fluctuations

Unbroken 1st order line to spatially inhomogeneous phases = “chiral spirals”

Hints in heavy ion data?

Fundamental problem in field theory: analogies to phase diagram for polymers
Could be CEP as well...

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**Quark-Gluon Plasma**

**Lifshitz regime**

**Hadronic**

**Chiral spirals**

**Quark matter**

$$T \uparrow$$

$$\mu \rightarrow$$
Lattice, hot QCD: no CEP at small $\mu$

Lattice: Hot QCD, 1701.04325

Expand about $\mu = 0$, power series in $\mu^{2n}$, $n = 1, 2, 3$.

Estimate radius of convergence. No sign of CEP by $\mu_{qk} \sim T$
Cluster expansion: no CEP at small $\mu$

Lattice: Vovchenko, Steinheimer, Philipsen & Stoecker, 1701.04325
Use cluster expansion method, different way of estimating power series in $\mu$
No sign of CEP by $\mu_{qk} \sim T$
So if there is no critical endpoint, what could be going on?
Lattice for $T = 0, \mu \neq 0$, two colors


Heavy pions, $m_\pi \sim 740$ MeV. $\sqrt{\sigma} = 470$ MeV. $32^4$ lattice, $a \sim .04$ fm

Confined until very high $\mu q \sim 1$ GeV. *Bare* Polyakov loop:

\[ \langle \text{loop} \rangle \uparrow \]
Lattice for $T = 0$, $\mu \neq 0$, two colors

Lattice: Bornyakov et al, 1711.01869.

String tension in time: nonzero up to $\mu_{qk} \sim 750$ MeV
Phases for $N_c = 2$, $T \sim 0$, $\mu \neq 0$

Braguta, Ilgenfritz, Kotov, Molochkov, & Nikolaev, 1605.04090 (earlier: Hands, Skellerud + …)

Lattice: $N_c = 2$, $N_f = 2$. $m_\pi \sim 400$ MeV, fixed $T \sim 50$ MeV, vary $\mu_{qk}$.

**Hadronic phase**: $0 \leq \mu_{qk} < m_\pi /2 \sim 200$ MeV. Confined, independent of $\mu$.

**Dilute baryons**: $200 < \mu_{qk} < 350$. Bose-Einstein condensate (BEC) of diquarks.

**Dense Baryons**: $350 < \mu_{qk} < 600$. Pressure *not* perturbative, BEC.

**Quarkyonic**: $600 < \mu_{qk} < 1100$: pressure $\sim$ perturbative, but excitations *confined* (Wilson loop $\sim$ area).

**Perturbative**: $1100 < \mu_{qk}$, but $\mu a$ too large.
Quarkyonic matter

McLerran & RDP 0706.2191

At large $N_c$, $g^2 N_c \sim 1$, $g^2 N_f \sim 1/N_c$, so need to go to large $\mu \sim N_c^{1/2}$. 

$$m_{\text{Debye}}^2 = g^2 \left( (N_c + N_f/2)T^2/3 + N_f \mu^2/(2\pi^2) \right)$$

Doubt large $N_c$ applicable at $N_c = 2$.

When does perturbation theory work?

$T = \mu = 0$: scattering processes computable for momentum $p > 1$ GeV

$T \neq 0$: $p > 2\pi T$, lowest Matsubara energy

$\mu \neq 0$, $T = 0$: $\mu$ is like a scattering scale, so perhaps $\mu_{\text{pert}} \sim 1$ GeV.

At least for the pressure. Excitations determined by region near Fermi surface
Possible phases of cold, dense quarks

Confined: $0 \leq \mu_{qk} < \frac{m_{\text{baryon}}}{3}$. $\mu$ doesn’t matter

Dilute baryons: $m_{\text{baryon}}/3 < \mu_{qk} < \mu_{\text{dilute}}$. Effective models of baryons, pions

Dense baryons: $\mu_{\text{dilute}} < \mu_{qk} < \mu_{\text{dense}}$. Pion/kaon condensates.

Quarkyonic: $\mu_{\text{dilute}} < \mu_{qk} < \mu_{\text{perturbative}}$. 1-dim. chiral spirals.

Perturbative: $\mu_{\text{perturbative}} < \mu_{qk}$. Color superconductivity

$\mu_{\text{perturbative}} \sim 1$ GeV?

Dense baryons and quarkyonic continuously related.

$U(1)$ order parameter in both.
Relevance for neutron stars

Fraga, Kurkela, & Vuorinen 1402.6618.

Maximum $\mu_{qk}$ may reach quarkyonic (for pressure), but true perturbative?

Ghisoiu, Gorda, Kurkela, Romatschke, Säppi, & Vuorinen, 1609.04339: $\text{pressure}(\mu_{qk}) \sim g^6$.

Will be able to compute $\Lambda_{\text{pert}} = \# \mu_{qk} \# \sim 1$?

\[
\mu_{q} \rightarrow \frac{m_B}{3} \rightarrow
\]

\[
\text{Dense Baryons} \quad \text{Quarkyonic} \quad \text{Perturbative}
\]
Quarkyonic matter: 1-dim. reduction

Kojo, Hidaka, McLerran & RDP 0912.3800: as toy model, assume confining potential

\[ \Delta_{00} = \frac{\sigma_0}{(\vec{p}^2)^2}, \quad \Delta_{ij} \sim \frac{1}{p^2} \]

Near the Fermi surface, reduces to effectively 1-dim. problem in patches. For *either* massless or massive quarks, excitations have zero energy about Fermi surface; just Fermi velocity \( v_F < 1 \) if \( m \neq 0 \).

Spin in 4-dim. -> “flavor” in 1-dim., so *extended* 2\( N_f \) flavor symmetry,

\[ \text{SU}(N_f)_L \times \text{SU}(N_f)_R \rightarrow \text{SU}(2N_f)_L \times \text{SU}(2N_f)_R \]. Similar to Glozman,1511.05857.

Extended 2\( N_f \) flavor sym. broken by transverse fluctuations, only approximate.

Number of patches \( N_{\text{patch}} \sim \mu/\sigma_0 \), so spherical Fermi surface recovered as \( \sigma_0 \rightarrow 0 \).
Transitions with # patches

Minimal number of patches = 6.

Probably occurs in dense baryonic phase.

In quarkyonic, presumably weak 1st order transitions as # patches changes.

Like Keplers....
Chiral spirals in 1+1 dimensions

In 1+1 dim., can eliminate $\mu$ by chiral rotation:

$$q' = e^{i\mu z \Gamma_5} q, \quad \overline{q}(\not{D} + i \mu \Gamma_0)q = \overline{q'} \not{D} q', \quad \Gamma_5 \Gamma_z = \Gamma_0$$

Thus a constant chiral condensate automatically becomes a chiral spiral:

$$\overline{q'} q' = \cos(2\mu z) \overline{q} q + i \sin(2\mu z) \overline{q} \gamma_5 q$$

Argument is only suggestive.

N.B.: anomaly ok, gives quark number: $\langle \overline{q} \Gamma_0 q \rangle = \mu / \pi$

Pairing is between quark & quark-hole, both at edge of Fermi sea.
Thus chiral condensate varies in $z$ as $\sim 2 \mu$. 
Bosonization in $1+1$ dimensions

Do not need detailed form of chiral spiral to determine excitations. Use bosonization. For one fermion,

$$\bar{\psi} \, \phi \, \psi \leftrightarrow (\partial_i \phi)^2$$

$\phi$ corresponds to $U(1)$ of baryon number. In general, non-Abelian bosonization. For flavor modes,

$$S_{\text{eff}}^{\text{flavor}} = \int dt \int dz \, 3 \frac{1}{16\pi} \text{tr}(\partial_\mu U^\dagger)(\partial_\mu U) + \ldots$$

where $U$ is a $\text{SU}(2 N_f)$ matrix.

Do not show Wess-Zumino-Witten terms for level $3 = \# $ colors.

Also effects of transverse fluctuations, reduce $\text{SU}(2 N_f) \rightarrow \text{SU}(N_f)$; quark mass

Lastly, $\text{SU}(3) + \text{level 2 } N_f$ sigma model. Modes are gapped by confinement.
Pion/kaon condensates & U(1) phonon

Overhauser ‘60, Migdal ‘71....Kaplan & Nelson ‘86...

Pion/kaon condensate:

\[ \langle \bar{q}_L q_R \rangle \sim \langle \Phi \rangle \sim \Phi_0 \exp(i(qz + \phi)t_3) \]

Condensate along \( \sigma \) and \( \pi^0 \) => \( t_3 \). Kaon condensate \( \sigma \) and K, etc.

Excitations are the SU\((N_f)\) Goldstone bosons and a “phonon”, \( \varphi \).

Phases with pion/kaon condensates and quarkyonic Chiral Spirals both spontaneously break U(1), have associated massless field.

*Continuously connected:* SU\((N_f)\) of \( \pi/K \) condensate => ~ SU\((2N_f)\) of CS’s.

Fluctuations same in both.

Perhaps WZW terms for \( \pi/K \) condensates?
Anisotropic fluctuations in Chiral Spirals

Spontaneous breaking of global symmetry =>
Goldstone Bosons have derivative interactions, $\sim \partial^2$

$\pi/K$ condensates and CS’s break both global and rotational symmetries

Interactions along condensate direction usual quadratic, $\sim \partial_z^2$

Those quadratic in transverse momenta, $\sim \partial_\perp^2$, cancel, leaving quartic, $\sim \partial_\perp^4$.

$$\mathcal{L}_{eff} = f_\pi^2 |(\partial_z - i k_0)U|^2 + \kappa |\partial_\perp^2 U|^2 + \ldots$$

Valid for both the U(1) phonon $\varphi$ and Goldstone bosons $U$

Hidaka, Kamikado, Kanazawa & Noumi 1505.00848;
Lee, Nakano, Tsue, Tatsumi & Friman, 1504.03185; Nitta, Sasaki & Yokokura 1706.02938
No long range order in Chiral Spirals

Consider tadpole diagram with anisotropic propagator

\[
\int d^2 k_\perp \, dk_z \, \frac{1}{(k_z - k_0)^2 + (k_\perp)^2} \sim \int d^2 k_\perp \, \frac{1}{k_\perp^2} \sim \log \Lambda_{IR}
\]

Old story for $\pi/K$ condensates: Kleinert ‘81; Baym, Friman, & Grinstein, ‘82.

Similar to smectic-C liquid crystals: ordering in one direction, liquid in transverse. Hence anisotropic propagator
Chiral Spirals in 1+1 dimensions

Overhauser/Migdal’s pion condensate: \((\sigma, \pi^0) = f_{\pi}(\cos(k_0 z), \sin(k_0 z))\)

Ubiquitous in 1+1 dimensions: Basar, Dunne & Thies, 0903.1868; Dunne & Thies 1309.2443+ ...

Wealth of exact solutions, phase diagrams at infinite \(N_f\).

Usual Gross-Neveu model:
- Phase diagram
- Chiral spiral:

\[\langle \bar{q}q \rangle \neq 0\]
\[\langle \bar{q}q \rangle_{CS} \neq 0\]
Chiral Spirals in 3+1 dimensions

In 3+1, *common* in NJL models: Nickel, 0902.1778 + ....Buballa & Carignano 1406.1367 + ...

In reduction to 1-dim, \( \Gamma_{5\text{-dim}} = \gamma_0 \gamma_z \), so chiral spiral between \( \bar{q}q \) & \( \bar{q}\gamma_0 \gamma_z \gamma_5 q \)
Both of these phase diagrams are dramatically affected by fluctuations:

no Lifshitz point in 1+1 or 3+1 dimensions at finite N

there is a *Lifshitz regime*
Standard phase diagram

\[ \mathcal{L} = (\partial_\mu \phi)^2 + m^2 \phi^2 + \lambda \phi^4 + \kappa \phi^6 \]

Negative quartic coupling, \( \lambda \), turns a 2nd order transition into 1st order. Two phases.

X = tri-critical point, \( m^2 = \lambda = 0 \)
Lifshitz phase diagram (in mean field theory)

\[ \mathcal{L}_{\text{Lifshitz}} = (\partial_0 \phi)^2 + Z(\partial_i \phi)^2 + \frac{1}{M^2}(\partial^2_i \phi)^2 + m^2 \phi^2 + \lambda \phi^4 \]

Negative kinetic term, \( Z < 0 \), generates spatially inhomogeneous phase, CS. Three phases.

\[ \langle \phi \rangle \neq 0 \quad \langle \phi \rangle = 0 \]

\[ X = \text{Lifshitz point, } m^2 = Z = 0 \]

\[ \langle \phi \rangle_{\text{CS}} \neq 0 \]

1st \( \uparrow \) \hspace{2cm} 2nd \( \uparrow \) \hspace{2cm} Z \uparrow \hspace{2cm} \langle \phi \rangle = 0 \]

\[ m^2 \rightarrow \]

\[ \langle \phi \rangle = 0 \]
No massless modes in too few dimensions

No massless modes in \( d \leq 2 \) dimensions:

\[
\int d^2 k \frac{1}{k^2} \sim \log \Lambda_{\text{IR}}
\]

Cannot break a continuous symmetry in \( d \leq 2 \) dimensions: instead of Goldstone bosons, generate a mass non-perturbatively.

**Lifshitz point:** \( Z = m^2 = 0 \), so propagator just \( \sim 1/k^4 \):

\[
\int d^4 k \frac{1}{k^4} \sim \log \Lambda_{\text{IR}}
\]

Hence *no* Lifshitz point in \( d \leq 4 \) (spatial) dimensions.

*Must* generate either a mass \( m^2 \), or term \( \sim Z \ p^2 \neq 0 \), non-perturbatively.
Lifshitz regime

Lifshitz regime (shaded):
$Z$ and/or $m^2$ are $\neq 0$ everywhere
strongly coupled, non-perturbative

$\langle \phi \rangle \neq 0$

$\langle \phi \rangle_{CS} \neq 0$

$Z \uparrow$

$\langle \phi \rangle = 0$

$2^{nd} \rightarrow$

$m^2 \rightarrow$

Brazovski 1st $\rightarrow$
Example: inhomogeneous polymers

Like mixing oil & water: polymers A & B, with AB diblock copolymer (“co-AB”)

Three phases: high temperature, A & B mix, symmetric phase

low temperature, little co-AB: A & B separate, broken phase

co-AB tends to decrease interface tension between A & B phases, can turn it negative. Like Z < 0

Low temperature, high concentration co-AB: “lamellar” phase, stripes of A & B. Like smectic.
Lifshitz point in inhomogenous polymers: mean field

Three phases, symmetric, broken, & spatially inhomogenous

Mean field predicts Lifshitz point at given T & concentration of co-AB

Lifshitz regime in inhomogenous polymers

Instead of Lifshitz point predicted by mean field theory, find

Bicontinuous microemulsion: $Z \neq 0, m^2 = 0$: Lifshitz regime

Jones & Lodge
Polymer Jour. 131 (44) 2012
Bicontinuous microemulsion: $Z \approx 0$

Experiment
Jones & Lodge,
Polymer Jour. 131 (44) 2012

Self-consistent field theory
Fredrickson, “The equilibrium theory of inhomogenous polymers”
Phase diagram for QCD in $T$ & $\mu$: usual picture

Two phases, one Critical End Point (CEP) between crossover and line of 1$^{\text{st}}$ order transitions

Ising fixed point, dominated by *massless* fluctuations at CEP
Lifshitz phase diagram for QCD

Lifshitz regime: strongly coupled, large fluctuations

Unbroken 1st order line to spatially inhomogeneous phases = “chiral spirals”

Heavy ions: could go through two 1st order transitions

$T_0$: maximum $T$, point of equal concentrations (unequal entropy)
Fluctuations at 7 GeV

Beam Energy Scan, down to 7 GeV.
Fluctuations *MUCH* larger when up to 2 GeV than to 0.8 GeV
Trivial multiplicity scaling? ... or Chiral Spiral?
But fluctuations in nucleons, not pions.

X. Luo & N. Xu, 1701.02105, fig. 37; Jowazee, 1708.03364

$$c_n = \frac{\partial^n}{\partial \mu^n} p(T, \mu)$$

- $c_4 / c_2$  \uparrow
- $\uparrow$ to .8 GeV
- $\uparrow$ to 2 GeV

$0$-$5\%$ Au+Au; $|y|<0.5$
- $0.4 < p_T < 0.8$ GeV (Published)
- $0.4 < p_T < 2.0$ GeV (Preliminary)

BES-II

STAR
Experimentally

For any sort of periodic structure (1D, 2D, 3D...),

Fluctuations concentrated about some characteristic momentum $k_0$

So “slice and dice”: bin in intervals, 0 to .5 GeV, .5 to 1., etc.

If peak in fluctuations in a bin not including zero, may be evidence for $k_0 \neq 0$.

Signals for Lifshitz regime?

Must measure fluctuations in pions, kaons...
NJL models and Lifshitz points

Consider Nambu-Jona-Lasino models.
Nickel, 0902.1778 & 0906.5295 + .... + Buballa & Carignano 1406.1367

\[ \mathcal{L}_{NJL} = \bar{\psi} (\partial + g \sigma) \psi + \sigma^2 \]

Integrating over \( \psi \),

\[ \text{tr} \log (\partial + g \sigma) \sim \ldots + \kappa_1 ((\partial \sigma)^2 + \sigma^4) + \ldots \]

Due to scaling, \( \partial \rightarrow \lambda \partial, \sigma \rightarrow \lambda \sigma \).
Consequently, in NJL @ 1-loop, tricritical = Lifshitz point.

Special to including only \( \sigma \) at one loop.
Not generic: violated by the inclusion of more fields, to two loop order, etc.

Improved gradient expansion near critical point:
Carignano, Anzuni, Benhar, & Mannarelli, 1711.08607.
Symmetric to CS: 1D (Brazovski) fluctuations

Consider \( m^2 > 0, Z < 0 \): minimum in propagator at non-zero momentum

Brazovski ‘75; Hohenberg & Swift ‘95 + ... ;
Lee, Nakano, Tsue, Tatsumi & Friman, 1504.03185; Yoshiike, Lee & Tatsumi 1702.01511

\[
\Delta^{-1} = m^2 + Z k^2 + k^4 / M^2 \\
= m_{\text{eff}}^2 - 2 Z k_z^2 + O(k_z^3, k_z k_{\perp}^2)
\]

\( k = (k_{\perp}, k_z - k_0) \): no terms in \( k_{\perp}^2 \), only \( (k_{\perp}^2)^2 \).

Due to spon. breaking of rotational sym.
1-loop tadpole diagram:

\[
\int d^3k \frac{1}{k_z^2 + m_{\text{eff}}^2 + \ldots} \sim M^2 \int \frac{dk_z}{k_z^2 + m_{\text{eff}}^2} \sim \frac{M^2}{m_{\text{eff}}}
\]

Effective reduction to 1-dim for any spatial dimension \( d \), any global symmetry
1\textsuperscript{st} order transition in 1-dim.

*Strong* infrared fluctuations in 1-dim., both in the mass:

\[
\Delta m^2 \sim \lambda \int d^3 k \frac{1}{k_z^2 + m_{\text{eff}}^2 + \ldots} \sim \lambda \frac{M}{m_{\text{eff}}}
\]

and for the coupling constant:

\[
\Delta \lambda \sim -\lambda^2 \int \frac{d^3 k}{(k_z^2 + m_{\text{eff}}^2 + \ldots)^2} \sim -\lambda^2 M^3 \int m_{\text{eff}}^{-1} \frac{dk_z}{k_z^4} \sim -\lambda \frac{M^3}{m_{\text{eff}}^3}
\]

Cannot tune \( m_{\text{eff}}^2 \) to 0: \( \lambda_{\text{eff}} \) goes negative, \textit{1\textsuperscript{st} order trans. induced by fluctuations}

\textit{Not} like other 1st order fluc-ind’ed trans’s: just that in 1-d, \( m_{\text{eff}}^2 \neq 0 \) always