Renormalization in Self-Consistent Approximation schemes at Finite Temperature III: Global Symmetries

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We investigate the symmetry properties for Baym's Φ-derivable schemes. We show that in general the solutions of the dynamical equations of motion, derived from approximations of the Φ-functional, do not fulfill the Ward-Takahashi identities of the symmetry of the underlying classical action, although the conservation laws for the expectation values of the corresponding Noether currents are fulfilled exactly for the approximation. Further we prove that one can define an effective action functional in terms of the self-consistent propagators which is invariant under the operation of the same symmetry group representation as the classical action. The requirements for this theorem to hold true are the same as for perturbative approximations: The symmetry has to be realized linearly on the fields and it must be free of anomalies, i.e., there should exist a symmetry conserving regularization scheme. In addition, if the theory is renormalizable in Dyson's narrow sense, it can be renormalized with counter terms which do not violate the symmetry.

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I. INTRODUCTION

Symmetry principles are of fundamental importance in physics. For instance the standard model of elementary particle physics rests on the principle of local gauge symmetries. These symmetries do not only guarantee the observed conservation laws but also ensure the physical consistency of the quantum field theory. Especially no unphysical degrees of freedom for the vector particles appear falsely in the particle spectrum. This ensures the causality and the positive definiteness of probabilities (in the vacuum case especially the unitarity of the S-Matrix).

Though fulfilled for the exact theory such symmetry concepts are commonly violated in most of the approximation schemes, in particular if partial resummations are performed. In this paper we address the question of global symmetries for Dyson resummation schemes based on the two-particle irreducible (2PI) action formalism [1, 2, 3] also known as Φ-derivable approximations [4]. Here Φ denotes the 2PI part of the generating functional Γ. As Baym could show such approximations have the property that expectation values of the Noether currents are exactly conserved and the solutions are thermodynamically consistent. These properties even survive a consistent gradient approximation of the Kadanoff-Baym equations leading to generalized quantum kinetic equations [5] which dynamically treat the full spectral functions of the particles and therefore permit a consistent transport treatment of resonances.

In the first paper of this series [6], in the following referred to as [I], we could verify an other important property of this type of approximations, namely that of renormalizability, provided the original theory is renormalizable in the standard sense. Using the path integral techniques [3] within the Schwinger-Keldysh real time formalism (for details and the notation conventions used here see [I]) closed equations for the renormalized self-consistent quantities with a counter term structure solely defined at the vacuum level could be established. First examples which include the fully self-consistent approximation at second order in the coupling, i.e., including the tadpole and the sunset self-energy diagrams, were given in the second part of this series [7], referred to as [II].

While for the solution to Φ-derivable approximations the expectation values of Noether currents are exactly conserved, this is no longer guaranteed at the correlator level or for higher order vertex functions. Thus in general the Ward-Takahashi identities (and thus also the symmetries of the underlying classical action) are violated already for the self-consistent self-energy. This was seen by Baym and Grinstein investigating various approximations for the $\mathrm{O}(N)$-model [8]. Considerations related to symmetries in approximation schemes were conducted already very early, e.g., in investigations of superfluid Helium within non-relativistic many-body theory[9]. At that time the notion of gap-less approximations was coined in the context of Nambu-Goldstone modes in spontaneously broken phases. In some recent attempts the same phenomena were investigated in [10] comparing various regularization and renormalization techniques, while in [11] the equations of motion were modified by introducing mean-field dependent effective vertex functions in the approximations at the gap-equation level.

In this paper we like to investigate the symmetry aspects of approximation schemes at a quite general level fully within the 2PI functional approach. In Sect. II we derive generalized Ward-Takahashi identities for the 2PI action functional. This is done in close analogy to the corresponding investigations for the usual perturbative quantum action functional (see for instance [12]). Thereby we prove the following properties of the 2PI action functional formalism: Provided the symmetry is realized as a linear representation on the fields and is free of anomalies the considerations show the following:

(1) If the 2PI action is truncated in a systematic way (loop expansion, $1/N$ expansion) it fulfills the same symmetries as the classical action if both, the classical fields and the Green's functions, are simultaneously transformed as tensors of the symmetry representation of 1^{st} and 2^{nd} rank, respectively.

(2) For the solutions of the corresponding self-consistent Dyson equation:

  (a) expectation values of Noether currents are conserved.

  (b) The scheme is void of double counting and fulfills detailed balance relations and is thus thermodynamically consistent.

  (c) However, Ward-Takahashi identities at the correlator level and for higher order vertex functions can be violated.

The latter problem is expected for such partial resummation schemes since from the point of view of the exact theory, which fulfills the symmetry, this expansion is of course incomplete. Diagrammatically it is easy to see that in general Φ-functional based partial Dyson resummations already violate the crossing symmetry of vertex functions in orders of the expansion parameter beyond that chosen for the truncation of the Φ-functional.

As an important result of this paper it will be shown in Sect. III how to overcome these problems. The strategy is that for any Φ-derivable approximation based on a truncated 2PI action $\Gamma[\varphi, G]$ one always can construct a non-perturbative approximation for the effective quantum action on top of the self-consistent solution which indeed recovers the symmetry in the above cited sense. This effective action, called $\tilde{\Gamma}[\varphi]$ is symmetric. It is expressed in terms of self-consistent propagators $\tilde{G}[\varphi]$ at a given mean field and provides a non-perturbative approximation to the usual 1PI quantum-action functional. This action was already considered in ref. [3] where the identity of vertex functions derived from Γ and $\tilde{\Gamma}$ has been shown for the exact case. Here we concentrate on the consequences of truncation schemes, where both pictures are no longer equivalent. Rather in terms of perturbative diagrams, the new action indeed supplements that minimal set of extra diagrams needed to recover the symmetry for the restricted set of diagrams resummed by the Φ-derivable Dyson scheme. The extra terms are encoded in a special Bethe-Salpeter equation and higher vertex equations which precisely cope with the chosen Φ-derivable scheme. Formally the functional derivatives of this effective action with respect to the background field, taken at the stationary point, define non-perturbative approximations to the self-energies and proper vertex functions which fulfill the usual Ward-Takahashi identities. We call the so generated vertex functions *external*, as they do not take part in the self-consistent scheme but are rather constructed a posteriori, once the self-consistent Dyson solutions are given. In this way symmetry-conserving non-perturbative approximations to these quantities are provided. It is shown that this also holds true for the renormalized physical quantities. Since the external vertex functions are defined as usual by multiple functional derivatives of the effective action all symmetries inherent in this functional like crossing symmetry are guaranteed for the solutions. We shall concentrate on the properties of the self-energy, which, if derived from $\tilde{\Gamma}$, respects the symmetries (e.g., posses Nambu-Goldstone modes). This general scheme, however, permits to repair the symmetry of any other higher order vertex functions such as the correlators of Noether currents $\langle j_\mu(x) j_\nu(y)\rangle$, which then are conserved in contrast to the ones given by the underlying self-consistent Dyson resummation.

In Sect. IV we discuss different approximation levels for the 2PI action applied to the $\mathrm{O}(N)$ linear σ-model. Besides the Hartree approximation (truncating the functional at the order λ) we also discuss the next approximation level, i.e., up to second order in the coupling. As the most simple example for the recovery of the $\mathrm{O}(N)$-symmetry violated by the Hartree approximation in the broken Nambu-Goldstone phase we show that the well known Random-Phase approximation (RPA), see, for instance, [13], is derivable from a non-perturbative effective action of the here defined kind. Some explicit numerical results are given for the chiral linear σ-model. This clarifies the question of symmetry violations of the Hartree approximation shown in [8] from the general point of view elaborated in this paper, which specifies the route to symmetry preserving vertex functions for any kind of Φ-derivable approximation. In the context of this series it is shown in sect. III C that a straight forward generalization of the renormalization procedure presented in the first part of this series [I] also leads to *closed renormalized* equations for these vertex functions at finite temperature which counter-term structures are solely defined on the vacuum level. The paper is closed with conclusion and outlook.

The appendices contain a formal derivation of the expression for the Φ-functional which uses the path-integral formalism developed in [3] and a short summary about Noether's theorem in the classical context which is needed in Sect. II to formulate the generalized Ward-Takahashi identities.

II. SYMMETRY PROPERTIES OF THE EXACT 2PI ACTION FUNCTIONAL

The 2PI action functional [1, 2, 4] can be introduced with help of the well known path integral methods [3]. Besides the standard local source J one also supplements a bilocal source B to the partition sum $Z = Z[J, B]$. The 2PI action functional $\Gamma[\varphi, G]$ is then defined as the double Legendre transformation of $W[J, B] = -\mathrm{i}\hbar \ln Z[J, B]$ with respect to both, J and B. Details and a slightly different derivation than in [3] are given in Appendix A. A formal loop expansion of the path integral yields

$$\Gamma[\varphi, G] = S[\varphi] + \frac{\mathrm{i}\hbar}{2} \mathrm{Tr} \ln(G_{12}^{-1}/M^2) + \frac{\mathrm{i}\hbar}{2} \int_{\mathcal{C}} \mathrm{d}(12)\, \mathscr{D}_{12}^{-1}(G_{12} - \mathscr{D}_{12}) + \Phi[\varphi, G]. \tag{1}$$

Here we have used the notation $\int_{\mathcal{C}} \mathrm{d}(1)\, f_1$ for the integration over d-dimensional space time in the sense of dimensional regularization and appropriate sums for the internal field components (i.e., "charge space" indices). Within the real-time formalism adapted to the equilibrium case the time integration runs along a modified Schwinger-Keldysh contour[1] which includes an imaginary branch of length $\mathrm{i}\beta$. In (1) $S[\varphi]$ denotes the classical action as a functional of the mean field φ, while the terms proportional to $\hbar$ account for the quantum fluctuations at the one loop level. The free propagator in the presence of the mean field φ is denoted by $\mathscr{D}$:

$$(\mathscr{D}^{-1})_{12} = \frac{\delta^2 S[\varphi]}{\delta\varphi_1 \delta\varphi_2}. \tag{2}$$

Finally Φ accounts for all higher order terms. It is diagrammatically defined as the sum of all *two-particle irreducible (2PI) closed diagrams* beyond the one-loop level which can be built with point-vertices defined by the interaction part of $S[\phi+\varphi]$, i.e., the part with at least three ϕ-fields expanded in the same way as the Feynman diagrams of perturbation theory. Thereby lines stand for *full propagators* G rather than free propagators. We shall give an analytical definition of the Φ functional which can be used for practical calculations of approximations like the coupling constant or loop expansion for the Φ-functional in appendix A.

The equations of motion for the mean field φ and the Green's function are given by the stationarity of Γ, i.e., at vanishing auxiliary sources $J = B = 0$ as

$$\frac{\delta\Gamma[\varphi, G]}{\delta\varphi} \stackrel{!}{=} 0, \quad \frac{\delta\Gamma[\varphi, G]}{\delta G} \stackrel{!}{=} 0. \tag{3}$$

Using Eq. (1) the second equation is seen as the Dyson equation for the full propagator

$$(\mathscr{D}^{-1})_{12} - (G^{-1})_{12} = 2\mathrm{i}\frac{\delta\Phi[\varphi, G]}{\delta G_{12}} := \Sigma_{12}. \tag{4}$$

Thus (3) gives a closed self-consistent set of equations of motion for the mean fields and the self-energies in terms of the exact Green's function G. The 2PI property of Φ avoids double counting in the sense that the diagrams of the self-energy with lines denoting exact Green's functions do not contain any self-energy insertion in any of its lines, i.e., it generates *one-particle irreducible (1PI) skeleton diagrams* for the self-energy by variation with respect to G. This is immediately clear from the fact that the derivative of Φ with respect to G diagrammatically implies to open anyone of its lines in the diagrams building Φ and taking the sum over the so obtained diagrams with two truncated external points.

For the discussion of the symmetry properties of the above defined functionals we take the $\mathrm{O}(N)$-symmetric ϕ^4-theory as an example. The generalization to other models and symmetries with more complicated field configurations is straight forward.

We use the fact that the path integral measure is invariant under a field translation $\vec{\phi}' = \vec{\phi} + \delta\vec{\phi}$. In first order of $\delta\vec{\phi}$ we find

$$\begin{split} 0 = \int \mathrm{D}\vec{\phi} \left[\int_{\mathcal{C}} \mathrm{d}(1) \left(\frac{\delta S}{\delta \phi_1^j} + J_{1j} \right) \delta\phi_1^j + \int_{\mathcal{C}} \mathrm{d}(12)\; B_{j1,k2} \phi_1^j \delta\phi_2^k \right] \\ \times \exp\left[\mathrm{i}S[\vec{\phi}] + \mathrm{i}\int_{\mathcal{C}} \mathrm{d}(1)\; J_{1j}\phi_1^j + \frac{\mathrm{i}}{2}\int_{\mathcal{C}} \mathrm{d}(12)\; B_{j1,k2}\phi_1^j \phi_2^k \right]. \end{split} \tag{5}$$

[1] besides the real-time parts along the real time axis from an initial time t_i to a final time t_f and back to t_i this contour also includes a branch parallel to the imaginary time axis from t_i to $t_i - \mathrm{i}\beta$ ($\beta = 1/T$ denotes the inverse temperature of the system). Finally we let the times t_i and t_f go to $-\infty$ and $+\infty$, respectively. All considerations can be extended to more general off-equilibrium initial statistical operators with the qualification that in this case it does not make sense to take $t_i \to -\infty$

Here subscripts and superscripts j, k denote the field components $(\phi^j) = \vec{\phi}$, and Einstein's summation convention is implied. For a *local* $\mathrm{O}(N)$-transformation

$$\delta\phi_1^j = \mathrm{i}\delta\chi_1^a (\tau^a)^j{}_{j'}\phi_1^{j'} \tag{6}$$

we obtain

$$\begin{aligned} 0 = \int \mathrm{D}\phi \int_{\mathcal{C}} \mathrm{d}(1) \left[\left(\frac{\delta S}{\delta\phi_1^j} + J_{1j} \right) \mathrm{i}(\tau^a)^j{}_{j'}\phi_1^{j'}\delta\chi_1^a + 2\int_{\mathcal{C}} \mathrm{d}(2)\ B_{j1,k2}\mathrm{i}(\tau^a)^j{}_{j'}\phi_1^{j'}\phi_2^k\delta\chi_1^a \right] \\ \times \exp\left[\mathrm{i}S[\vec{\phi}] + \mathrm{i}\int_{\mathcal{C}} \mathrm{d}(1')\ J_{1'j'}\phi_{1'}^{j'} + \frac{\mathrm{i}}{2}\int_{\mathcal{C}} \mathrm{d}(1'2')\ B_{1'j',2'k'}\phi_{1'}^{j'}\phi_{2'}^{k'} \right]. \end{aligned} \tag{7}$$

Since the classical action functional is invariant under *global* $\mathrm{O}(N)$ transformations (6), c.f. Appendix A, we read off from (B8) that (7) can be expressed in the form

$$\begin{aligned} 0 = \int_{\mathcal{C}} \mathrm{d}(1)\ \delta\chi_1^a \left\{ \partial_\mu j_1^{a\mu}\left[\frac{\delta}{\mathrm{i}\delta\vec{J}}\right] Z[\vec{J},B] + J_{1j}\mathrm{i}(\tau^a)^j{}_{j'}\frac{\delta Z[\vec{J},B]}{\delta \mathrm{i}J_{1j'}} \right\} \\ + 2\int_{\mathcal{C}} \mathrm{d}(12)\ \delta\chi_1^a B_{1j,2k}\mathrm{i}(\tau^a)^j{}_{j'}\frac{\delta Z[\vec{J},B]}{\delta \mathrm{i}B_{1j',2k}}. \end{aligned} \tag{8}$$

Since $\delta\chi_1^i$ is an arbitrary function this can be brought to the local expression

$$0 = \partial_\mu j_1^{a\mu}\left[\frac{\delta}{\mathrm{i}\delta\vec{J}}\right] Z[\vec{J},B] + J_{1j}\mathrm{i}(\tau^a)^j{}_{j'}\frac{\delta Z[\vec{J},B]}{\delta \mathrm{i}J_{1j'}} + 2\int_{\mathcal{C}} \mathrm{d}(2)\ B_{1j,2k}\mathrm{i}(\tau^a)^j{}_{j'}\frac{\delta Z[\vec{J},B]}{\delta \mathrm{i}B_{1j',2k}}. \tag{9}$$

For the solution of the equations of motion we have $\vec{J} = 0$ and $B = 0$. Thus the expectation value of the current is conserved as we read off from (9):

$$\partial_\mu j_1^{a\mu}\left[\frac{\delta}{\mathrm{i}\delta\vec{J}}\right] Z[\vec{J},B] = Z[\vec{J},B]\partial_\mu \left\langle \mathbf{j}_1^{a\mu} \right\rangle = 0. \tag{10}$$

Here the expectation value has to be read as the quantum statistical expectation value for the local current operator if interpreted within the operator formalism of quantum field theory. Here and in the following we write operators as bold face upright symbols.

For sake of completeness we write down the local Ward-Takahashi identities (WTIs) for the current. From (B11) we find for the $\mathrm{O}(N)$-Noether current the expression

$$j_{1\mu}^a[\phi] = -\mathrm{i}(\tau^a)_{jj'}\phi_1^j\partial_\mu\phi_1^{j'}. \tag{11}$$

Using this in (9) yields the local WTIs for the functional $W = -\mathrm{i}\ln Z$:

$$-2\left[\Box^{(2)}\frac{\delta W}{\delta B_{1k,2k'}}\right]_{1=2}(\tau^a)_{kk'} + \vec{J}_{1j}(\tau^a)^j{}_{j'}\frac{\delta W}{\delta J_{1j'}} + 2\int_{\mathcal{C}} \mathrm{d}(2)\ B_{j1,k2}(\tau^a)^j{}_{j'}\frac{\delta W}{\delta B_{j'1,k2}} = 0. \tag{12}$$

Finally we also obtain an expression for the Γ-functional by using the relations (A5) and (A15):

$$\begin{aligned} -\mathrm{i}\partial_\mu \left\langle \mathbf{j}_1^\mu[\boldsymbol{\phi}] \right\rangle &= \left(-\varphi_1^k\Box\varphi_1^{k'} + 2\mathrm{i}\left[\Box^{(2)}G_{1k,2k'}\right]_{1=2} \right)(\tau^a)_{kk'} \\ &= \frac{\delta\Gamma}{\delta\vec{\varphi}_1}\hat{\tau}^a\vec{\varphi}_1 + 2\int_{\mathcal{C}} \mathrm{d}(2)\ \frac{\delta\Gamma}{\delta G_{j1,k2}}\tau_{jj'}^a G_{1j',2k}. \end{aligned} \tag{13}$$

Again it follows immediately that for the solutions of the equations of motion (3) the expectation value of the O(N)-Noether current is conserved. Taking further derivatives of (9) yields WTIs for higher order Green's functions of the Noether current, which we do not need in the further line of arguments.

For the further symmetry analysis we like to concentrate on properties of the 2PI functional (1). We thus go back to (7) and set $\delta\chi^i = \text{const}$. Then the first term in the curly bracket is a complete divergence and thus the integral vanishes and due to the linear independence of the constants $\delta\chi^i$ we obtain the *global form* of the generalized *Ward-Takahashi identities*.

Especially this proves that the generating functional Z is invariant under *global* O(N) transformations when the local source J_{1j} is transformed in the contragredient way of the fields and the bilocal sources $B_{1j,2k}$ as a covariant tensor of 2^{nd} rank. Since (8) is of first order in the derivatives of Z with respect to the sources the same holds true for W:

$$\int_{\mathcal{C}} \mathrm{d}(1)\; J_{1j}\mathrm{i}(\tau^a)^j{}_{j'} \frac{\delta W[\vec{J},B]}{\delta J_{1j'}} + \int_{\mathcal{C}} \mathrm{d}(12)\; \left[B_{1j',2k}(\tau^a)^{j'}{}_j + B_{1j,2k'}(\tau^a)^{k'}{}_k\right] \frac{\delta W[\vec{J},B]}{\delta B_{j1,k2}}, \tag{14}$$

and from (A5) we find (again for *global* transformations):

$$\int_{\mathcal{C}} \mathrm{d}(1)\; \frac{\delta\Gamma[\vec{\varphi},G]}{\delta\vec{\varphi}_1}\hat{\tau}^a\vec{\varphi}_1 + \int_{\mathcal{C}} \mathrm{d}(12)\; \frac{\delta\Gamma[\vec{\varphi},G]}{\delta G_{12}^{jk}} \left[(\tau^a)^j{}_{j'}G_{12}^{j'k} + (\tau^a)^k{}_{k'}G_{12}^{jk'}\right] = 0. \tag{15}$$

This result can be derived also directly from (13) by taking the integral $\int_{\mathcal{C}} \mathrm{d}(1)\; \ldots$ on both sides of the equation.

III. SYMMETRIES OF 2PI Φ-DERIVABLE APPROXIMATIONS

So far the Ward-identities were derived for the *exact functionals, Green's functions and mean fields.* In particular it is important to realize that the identity

$$\mathrm{i}G_{12}^{jk} := -\mathrm{i}\frac{\delta^2 W[J,B]}{\delta J_{1j}\delta J_{2k}} = 2\frac{\delta W[J,B]}{\delta B_{1j,2k}} - \varphi_1^j\varphi_2^k \text{ for the } \textit{exact functional} \tag{16}$$

was proven using the underlying path integral definition of the exact functionals. We now step towards the Φ-derivable approximations. They are defined by a *truncation* of the auxiliary functional $\Phi[\varphi,G]$ in the definition (1) of the generating functional $\Gamma[\varphi,G]$ while keeping the variational properties (3) which define the equations of motion for mean field and propagator

$$\frac{\delta\Gamma_{\text{apprx}}[\varphi,G]}{\delta\varphi} \stackrel{!}{=} 0, \quad \frac{\delta\Gamma_{\text{apprx}}[\varphi,G]}{\delta G} \stackrel{!}{=} 0. \tag{17}$$

The latter equation of motion defines the self-energy in terms of the self-consistent propagator by

$$\Sigma_{12} := (\mathscr{D}^{-1})_{12} - (G^{-1})_{12} = 2\mathrm{i}\frac{\delta\Phi_{\text{apprx}}[\varphi,G]}{\delta G_{12}}. \tag{18}$$

Thereby $\Phi[\varphi,G]$ can be truncated according to various schemes, like expansion in powers of λ, in loop order or in powers of $1/N$ (see [14]), excluding the internal structure of the Green's functions from the counting. The sole requirement for the thus constructed Φ_{apprx} is that it remains *invariant* under the symmetry transformations as explained in appendix A, i.e., that the generalized WTIs (15) hold true also for the approximation. In our case of the linear O(N)-model this is the case for the just mentioned schemes. The preferred choice may be motivated by the physical problem.

Since in the so defined 2PI approximation one has a functional structure solely defined in terms of the approximated one- and two-point functions φ and G, path integral properties and relations such as (16) do not need to hold true any longer. Thus irrespective of the chosen truncation scheme the n-point functions defined by the approximated 2PI functional generally may no longer coincide with the corresponding 1PI vertex-functions and thus WTIs derived from the 1PI formalism may be violated for those self-consistent approximations.

The reason is simply seen in diagrammatic terms. The Schwinger-Dyson resummation creates a very restricted subset of 1PI diagrams resulting from the iterative insertion of self-energy pieces. In particular this implies that already the crossing symmetry is violated at orders of the respective expansion parameter beyond those included in Φ_{apprx}. Thus it has to be expected that in general for the self-consistent approximation scheme (17) the WTIs for the vertex functions derived from the 1PI formalism are not fulfilled for the self-consistent approximations of the self-energy and higher n-point functions derived from the 2PI functional. For example the symmetries may already be violated for the self-energy Σ. Especially for systems in the Nambu-Goldstone phase this implies that the self-consistent propagators may not comply with Goldstone's theorem.

In order to cure this problem we supplement the 2PI approximation scheme (17) by an additional effective action $\tilde{\Gamma}_{\text{apprx}}[\varphi]$ defined with respect to the self-consistent solution as

$$\begin{aligned} &\tilde{\Gamma}_{\text{apprx}}[\varphi] = \Gamma_{\text{apprx}}[\varphi, \tilde{G}[\varphi]] \text{ with } \tilde{G}[\varphi] \text{ defined by} \\ &\left.\frac{\delta\Gamma_{\text{apprx}}[\varphi, G]}{\delta G}\right|_{G=\tilde{G}[\varphi]} \equiv 0. \end{aligned} \tag{19}$$

Here $\tilde{G}[\varphi]$ is given as the Schwinger-Dyson solution in presence of a given mean field φ. Strategies like this date back to Baym and Kadanoff [15], the equivalence of both functionals $\Gamma[\varphi, G]$ and $\tilde{\Gamma}[\varphi]$ at the exact level was shown in ref. [3], while the consequences for truncation schemes were discussed, e.g., in [16] in the context of a background field formulation. Here we will show that independent of the chosen trancation scheme $\tilde{\Gamma}_{\text{apprx}}[\varphi]$ permits to construct proper vertex functions on top of the self-consistent solutions of (17), which then obey the symmetries as described by the usual 1PI WTIs. In diagrammatic terms, to any Φ-derivable approximation this procedure supplements that minimal set of diagrams which is needed to recover the symmetries. Effectively φ can be considered as a background field and $\tilde{\Gamma}_{\text{apprx}}[\varphi]$ permits to construct the linear response of the system with respect to a fluctuation in φ around the self-consistent solution of the equations of motion (17). The latter coincides with the stationary point $\tilde{\varphi}$ of the effective action functional $\tilde{\Gamma}_{\text{apprx}}[\varphi]$

$$\begin{aligned} \left.\frac{\delta\tilde{\Gamma}_{\text{apprx}}[\vec{\varphi}]}{\delta\vec{\varphi}}\right|_{\vec{\varphi}=\tilde{\varphi}} &= \left(\frac{\delta\Gamma_{\text{apprx}}[\vec{\varphi}, G]}{\delta\vec{\varphi}} + \int_{\mathcal{C}} \mathrm{d}(12)\, \frac{\delta\Gamma_{\text{apprx}}[\vec{\varphi}, G]}{\delta G_{12}^{jk}} \frac{\delta\tilde{G}_{12}^{jk}[\vec{\varphi}]}{\delta\vec{\varphi}}\right)_{G=\tilde{G}[\vec{\varphi}], \vec{\varphi}=\tilde{\varphi}} \\ &= \left(\frac{\delta\Gamma_{\text{apprx}}[\vec{\varphi}, G]}{\delta\vec{\varphi}}\right)_{G=\tilde{G}[\vec{\varphi}], \vec{\varphi}=\tilde{\varphi}} \stackrel{!}{=} 0, \end{aligned} \tag{20}$$

i.e., $\tilde{\varphi}$ and $\tilde{G}[\tilde{\varphi}]$ solve (17).

Now since the derivation of the WTIs (15) only relies on the symmetry of the $\Gamma_{\text{apprx}}[\varphi, G]$-functional under the $\mathrm{O}(N)$-transformations of φ and G as a contravariant vector and a contravariant tensor of 2^{nd} rank, respectively, and this by construction holds true for the approximated functional, relation (15) is also valid for the approximation. Therefore we immediately conclude that the *non-perturbative effective action functional* (19) is $\mathrm{O}(N)$ symmetric in the sense of the classical action, i.e.,

$$\int_{\mathcal{C}} \mathrm{d}(1)\, \frac{\delta\tilde{\Gamma}_{\text{apprx}}[\vec{\varphi}]}{\delta\varphi_1^j} (\tau^a)^j{}_{j'} \varphi_1^{j'} = 0. \tag{21}$$

Thus, if we define self-energies and higher proper vertex functions by the usual definition as multiple derivatives of $\tilde{\Gamma}$, then these functions fulfill the *usual* 1PI WTIs, and thus obey all symmetry properties derived from them. We call the so constructed functions *external*, as they do not take part in the self-consistent scheme but rather are calculated as a function of the frozen self-consistent solutions of (17), the latter being referred to as *internal*. While external and internal quantities coincide at the exact level, this is commonly no longer the case for the here discussed approximation schemes based on a truncated Φ-functional. Below we drop the label "apprx".

A. Goldstone's Theorem

As a first quantity we discuss the *external propagator* defined from $\tilde{\Gamma}$ by

$$(G_{\text{ext}}^{-1})_{1j,2k} = \left. \frac{\delta^2 \tilde{\Gamma}[\varphi]}{\delta \varphi_1^j \delta \varphi_2^k} \right|_{\varphi=\tilde{\varphi}}, \tag{22}$$

which fulfills the WTI

$$\int_{\mathcal{C}} \mathrm{d}(1)\; (G_{\text{ext}}^{-1})_{j1,k2} (\tau^a)^j{}_{j'} \varphi_1^{j'} = 0, \tag{23}$$

as can be immediately seen by taking the functional derivative of (21) with respect to φ_2^k and using the fact that $\tilde{\Gamma}$ is stationary for $\varphi = \tilde{\varphi}$. For a translationally invariant situation (for instance for the vacuum or thermal equilibrium) this can easily be Fourier transformed leading to

$$(G_{\text{ext}}^{-1})_{jk}(p=0)(\tau^a)^j{}_{j'}\varphi^{j'} = -(M^2)_{jk}(\tau^a)^j{}_{j'}\varphi^{j'} = 0, \tag{24}$$

where M^2 denotes the (thermal) mass matrix for the field degrees of freedom. This reflects Goldstone's theorem: There are as many massless states as group generators which do not annihilate the mean field, i.e., the number of massless Goldstone bosons is the dimension of the symmetry group minus the dimension of the symmetry group of the mean field. In our case the symmetry group is $\mathrm{O}(N)$ and the symmetry group of the mean field is $\mathrm{O}(N-1)$, which means that we have $(N-1)N/2 - (N-2)(N-1)/2 = N-1$ massless Goldstone Bosons in the spontaneously broken symmetry phase.

B. The external propagator

Now we explicitly construct the external propagator. From (19) we find

$$\left(G_{\text{ext}}^{-1}\right)_{1j,2k} = \left[\frac{\delta^2 \Gamma[\varphi, G]}{\delta \varphi_1^j \delta \varphi_2^k} + \int_{\mathcal{C}} \mathrm{d}(3'4')\, \frac{\delta^2 \Gamma[\varphi, G]}{\delta \varphi_1^j \delta G_{3'4'}^{j'k'}} \frac{\delta \tilde{G}_{3'4'}^{j'k'}}{\delta \varphi_2^k} \right]_{G=\tilde{G}[\tilde{\varphi}],\varphi=\tilde{\varphi}}. \tag{25}$$

The derivative of $\tilde{G}$ with respect to the background field φ can be expressed through the identity

$$\int_{\mathcal{C}} \mathrm{d}(2')\; (\tilde{G}^{-1})_{1j,2'k'} \tilde{G}_{2'2}^{k'k} = \delta_{12}^{(d)} \delta_j^k. \tag{26}$$

Taking its derivative with respect to the mean field yields

$$\int_{\mathcal{C}} \mathrm{d}(2') \left[\frac{\delta (\tilde{G}^{-1})_{1j,2'k'}}{\delta \varphi_3^l} \tilde{G}_{2'2}^{k'k} + (\tilde{G}^{-1})_{1j,2'k'} \frac{\delta \tilde{G}_{2'2}^{k'k}}{\delta \varphi_3^l} \right] = 0, \tag{27}$$

implying

$$\frac{\delta G^{jk}_{12}}{\delta\varphi^l_3} = -\int_{\mathcal{C}} \mathrm{d}(1'2')\, \frac{\delta\tilde{G}^{-1}_{1'j',2'k'}}{\delta\varphi^l_3}\tilde{G}^{jj'}_{11'}\tilde{G}^{kk'}_{22'}. \tag{28}$$

The functional derivative of $\tilde{G}^{-1}$, which is a three-point function

$$\Lambda^{(3)}_{1j,2k;3l} = \frac{\delta\tilde{G}^{-1}_{1j,2k}}{\delta\varphi^l_3}, \tag{29}$$

can be implicitly expressed in form of a *Bethe-Salpeter equation* (BS)

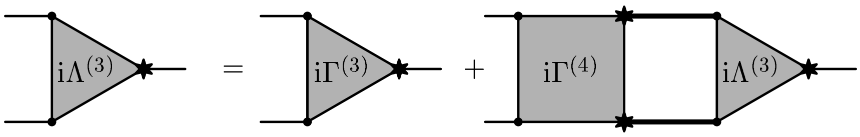
$$\Lambda^{(3)}_{1j,2k;3l} = \Gamma^{(3)}_{1j,2k;3l} - \mathrm{i}\int_{\mathcal{C}} \mathrm{d}(3'4'3''4'')\, \Gamma^{(4)}_{1j,2k;3'l',4'm'}\tilde{G}^{l'l''}_{3'3''}\tilde{G}^{m'm''}_{4'4''}\Lambda^{(3)}_{3''l'',4''m'';3l}. \tag{30}$$

We denote the resummed, i.e., two-particle reducible n-point vertex functions by $\Lambda^{(n)}$ while the corresponding irreducible parts are denoted by $\Gamma^{(n)}$. Constructed through Φ the latter

$$\begin{aligned} \Gamma^{(3)}_{1j,2k;3l} &= \left[\frac{\delta^3 S[\varphi]}{\delta\varphi^j_1\delta\varphi^k_2\delta\varphi^l_3} - 2\mathrm{i}\frac{\delta^2\Phi[\varphi,G]}{\delta G^{jk}_{12}\delta\varphi^l_3}\right]_{G=\tilde{G}[\tilde{\varphi}],\varphi=\tilde{\varphi}}, \\ \Gamma^{(4)}_{1j,2k;3l,4m} &= -2\left[\frac{\delta^2\Phi[\varphi,G]}{\delta G^{jk}_{12}\delta G^{lm}_{34}}\right]_{G=\tilde{G}[\tilde{\varphi}],\varphi=\tilde{\varphi}} \end{aligned} \tag{31}$$

are functionals of the self-consistent propagator $\tilde{G}$.

In terms of diagrams the BS-equation (30) can be depicted as follows:

$\mathrm{i}\Lambda^{(3)} =$ 

. (32)

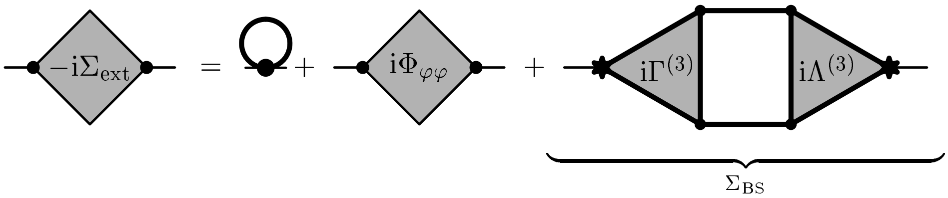
The different external points of the vertex functions, dots and stars, indicate the different kind of points, which are separated by a semicolon in (30). This specialty of those diagrams clearly shows that only certain channels are resummed namely those which are not contained in the self-consistent scheme. As explained in [I] the Dyson resummation implies s-channel ladder resummations, while the here given BS-equation creates t- and u-channel ladders, this way recovering crossing and all global symmetries. This again shows the virtue of the Φ-functional method. Through the effective action functional $\tilde{\Gamma}[\varphi]$ it permits the construction of fully symmetry preserving vertex functions void of double counting, although each dynamical equation, the Dyson and BS-equation, for themselves violate these symmetries. The kernel $\Gamma^{(4)}$ of the BS-equation is symmetric with respect to the simultaneous change of the pairs 1,2 with 3,4 and 2PI for separating these pairs. In addition the Φ-functional ensures that especially the crossing symmetry is also recovered for the counter terms, a property necessary to renormalize the proper n-point functions.[2] This is non-trivial since the counter terms appear to arbitrary order in the sense of an $\hbar$- or coupling constant expansion.

[2] Power counting and Weinberg's convergence theorem show that in our case the superficial degree of divergence of a diagram γ is $\delta(\gamma) = 4 - E$, where E is the number of external legs of the diagram. This means that due to the BPHZ-renormalization procedure only proper n-point functions with $n \leq 4$ have to be renormalized.

Using (30), (31), (A15), and (A4) in (25) we finally obtain for the external self-energy

$$
\begin{aligned}
(\Sigma_{\mathrm{ext}})_{1j,2k} = -\Bigg[& \frac{\mathrm{i}}{2}\int_{\mathcal{C}} \mathrm{d}(1'2')\, \frac{\delta^4 S[\varphi]}{\delta\varphi_1^j \delta\varphi_2^k \delta\varphi_{1'}^{j'} \delta\varphi_{2'}^{k'}} G_{1'2'}^{j'k'} + \frac{\delta^2 \Phi[\varphi, G]}{\delta\varphi_1^j \delta\varphi_2^k} \\
& - \frac{\mathrm{i}}{2}\int_{\mathcal{C}} \mathrm{d}(3'4'3''4'')\, \Gamma^{(3)}_{3'j',4'k';1j} G^{j'j''}_{3'3''} G^{k'k''}_{4'4''} \Lambda^{(3)}_{3''j'',4''k'';2k} \Bigg]_{G=\tilde{G}[\tilde{\varphi}],\varphi=\tilde{\varphi}}
\end{aligned}
\tag{33}
$$

Again this can be easily expressed with help of diagrams



, (34)

where the last term of the external self-energy, the BS-part Σ_{BS}, needs special care with respect to renormalization. The external self-energy Σ_{ext}, here derived for a general Φ-derivable truncation scheme is frequently called the mass matrix, as it is used to determine the masses of the fluctuations in the phase of broken symmetry.

C. Solution and renormalization of the Bethe-Salpeter equation

The BS-equation now given in short hand four-point function notation

$$\Lambda^{(3)} = \Gamma^{(3)} + \Gamma^{(4)} G^{(2)} \Lambda^{(3)}, \text{ where } G^{(2)}_{12;34} = -\mathrm{i} G_{13} G_{24} \tag{35}$$

can be solved in terms of the four-point function $\Lambda^{(4)}$ as

$$\Lambda^{(3)} = \Gamma^{(3)} + \Lambda^{(4)} G^{(2)} \Gamma^{(3)}, \tag{36}$$

where the four-point function $\Lambda^{(4)}$ is defined through two equivalent BS-equations

$$
\begin{aligned}
\Lambda^{(4)} &= \Gamma^{(4)} + \Gamma^{(4)} G^{(2)} \Lambda^{(4)} &(37)\\
&= \Gamma^{(4)} + \Lambda^{(4)} G^{(2)} \Gamma^{(4)}. &(38)
\end{aligned}
$$

The latter four-point BS-equation is identical to that already considered in the first paper [I] of this series, however with two differences: In I only s-channel resummations were considered which restricts the momentum arguments to forward scattering, and secondly the BS-equation was used solely at the vacuum level. Here, however, the BS-equation truly acts at finite temperature. The latter implies that for the renormalization procedure vacuum and finite temperature pieces need to be separated along the lines given in [I]. Again however the only subdiagrams to be renormalized are the four-point functions which lead to the renormalized vacuum function Λ defined in [I].

Considering the difference between the four-point BS-equation at finite T and that at vacuum leads to the following expression[3]

$$
\begin{aligned}
\Lambda^{(4)} - \Lambda^{(4;\mathrm{vac})} &= \Gamma^{(4)} - \Gamma^{(4;\mathrm{vac})} + \Gamma^{(4)} G^{(2)} \Lambda^{(4)} - \Gamma^{(4;\mathrm{vac})} G^{(2,\mathrm{vac})} \Lambda^{(4;\mathrm{vac})} \\
&= \Gamma^{(4)} - \Gamma^{(4;\mathrm{vac})} + \Lambda^{(4)} G^{(2)} (\Gamma^{(4)} - \Gamma^{(4;\mathrm{vac})}) \\
&\quad + \Lambda^{(4)} G^{(2)} \Gamma^{(4;\mathrm{vac})} - \Gamma^{(4;\mathrm{vac})} G^{(2,\mathrm{vac})} \Lambda^{(4;\mathrm{vac})} \\
&= \Gamma^{(4)} - \Gamma^{(4;\mathrm{vac})} + \Lambda^{(4)} G^{(2)} (\Gamma^{(4)} - \Gamma^{(4;\mathrm{vac})}) \\
&\quad + (\Gamma^{(4)} - \Gamma^{(4;\mathrm{vac})}) G^{(2,\mathrm{vac})} \Lambda^{(4;\mathrm{vac})} \\
&\quad + \Lambda^{(4)} G^{(2)} (\underbrace{\Lambda^{(4;\mathrm{vac})} - \Gamma^{(4;\mathrm{vac})} G^{(2,\mathrm{vac})} \Lambda^{(4;\mathrm{vac})}}_{\Gamma^{(4;\mathrm{vac})}}) \\
&\quad - (\underbrace{\Lambda^{(4)} - \Lambda^{(4)} G^{(2)} \Gamma^{(4)}}_{\Gamma^{(4)}}) G^{(2,\mathrm{vac})} \Lambda^{(4;\mathrm{vac})} \\
&= \Gamma^{(4)} - \Gamma^{(4;\mathrm{vac})} + \Lambda^{(4)} G^{(2)} (\Gamma^{(4)} - \Gamma^{(4;\mathrm{vac})}) \\
&\quad + (\Gamma^{(4)} - \Gamma^{(4;\mathrm{vac})}) G^{(2,\mathrm{vac})} \Lambda^{(4;\mathrm{vac})} \\
&\quad + \Lambda^{(4)} (G^{(2)} - G^{(2,\mathrm{vac})}) \Lambda^{(4;\mathrm{vac})} \\
&\quad + \Lambda^{(4)} G^{(2)} (\Gamma^{(4)} - \Gamma^{(4;\mathrm{vac})}) G^{(2,\mathrm{vac})} \Lambda^{(4;\mathrm{vac})}
\end{aligned}
\tag{39}
$$

Since the $\Gamma^{(4)}$-functions are 2PI when cutting the corresponding diagrams such that the two external point pairs become separated, the subtractions cause the explicite loops in the final expression to be finite such that only the vacuum quantities are to be renormalized.

This vacuum renormalization is obtained with the same techniques as explained in [I]. In the case of a spontaneously broken symmetry only the renormalization description has to be changed, since the on-shell scheme described in [I] would lead to artificial infrared singularities. Rather here a "mass-independent renormalization scheme" as the MS or $\overline{\mathrm{MS}}$ scheme in dimensionally regularized perturbation theory is required. As generally shown in [I], for self-consistent schemes BPHZ-like renormalizations are more convenient to use, especially in numerical simulations as presented in [II]. Thus we introduce a mass renormalization scale $\tilde{\mu}$, i.e., *for the counter terms* we set the mass parameter m^2 to $\tilde{\mu}^2$. Since the UV divergences are ruled by the asymptotic behavior of the Green's functions at large loop momenta, the same counter terms also render the integrals finite for the spontaneously broken phase where $m^2 = -\tilde{m}^2 < 0$.

Then we can apply the same renormalization conditions as in [I] for the symmetric phase now however taken at $m^2 = \tilde{\mu}^2$:

$$
\begin{aligned}
\Sigma^{(\mathrm{vac})}(p^2 = 0; m^2 = \tilde{\mu}^2) &= 0, \\
\partial_{p^2} \Sigma^{(\mathrm{vac})}(p^2; m^2 = \tilde{\mu}^2)|_{p^2=0} &= 0, \\
\partial_{m^2} \Sigma^{(\mathrm{vac})}(p^2; m^2 = \tilde{\mu}^2)|_{p^2=0} &= 0, \\
\Lambda^{(4;\mathrm{vac})}(s = t = u = 0; m^2 = \tilde{\mu}^2) = \Gamma^{(4;\mathrm{vac})}(s = t = u = 0; m^2 = \tilde{\mu}^2) &= \frac{\lambda}{2}.
\end{aligned}
\tag{40}
$$

Herein s, t and u are the usual Mandelstam variables for two-particle scattering kinematics. This procedure shows that those counter-terms necessary to render the self-consistent Dyson-equation of motion finite also enter here for the BS-equation. Especially the crossing symmetry of counterterms is also recovered consistently with the chosen approximation for Φ.

[3] As explained in detail in [I], in this subtraction technique all vacuum functions are contour diagonal, i.e., they vanish for arguments with mixed vertex placement

Given the renormalized $\Lambda^{(4;\text{vac})}$, Eq. (39) constitutes a regular integral equation to determine the finite T-dependent function $\Lambda^{(4)}$. The steps towards $\Lambda^{(3)}$ and finally Σ^{BS} only involve to close a further loop which is rendered UV-finite considering the respective differences

$$\begin{aligned}\Lambda^{(3)} - \Lambda^{(3;\text{vac})} =& \Gamma^{(3)} - \Gamma^{(3;\text{vac})} + \underbrace{\Lambda^{(4)}\tilde{G}^{(2)}\Gamma^{(3)} - \Lambda^{(4;\text{vac})}G^{(2,\text{vac})}\Gamma^{(3;\text{vac})}}_{\text{UV-finite}} \\ \Sigma_{\text{BS}} - \Sigma_{\text{BS}}^{\text{vac}} =& \underbrace{\Gamma^{(3)}\tilde{G}^{(2)}\Lambda^{(3)} - \Gamma^{(3;\text{vac})}G^{(2;\text{vac})}\Lambda^{(3;\text{vac})}}_{\text{UV-finite}}.\end{aligned} \tag{41}$$

Here it is important to notice that in purely scalar theories without derivative couplings three-point vertices are only logarithmically divergent. Further due to field-reflection symmetry of a theory with no generic three-point interaction term in the classical action no such counter term is necessary to render the three-point functions $\Gamma^{(3)}$ and $\Lambda^{(3)}$ finite; only counter terms for four-point functions appear in the renormalization scheme cf. (41).

This completes the proof that also for the non-perturbative higher order vertex functions resulting from $\tilde{\Gamma}$ the renormalization parts can entirely be defined at the vacuum level.

D. Comments

The main result of our symmetry analysis can be summarized as follows: Truncated expansions of the 2PI *functional* $\Gamma[\vec{\varphi}, G]$ with respect to parameters, like $\hbar$ or $1/N$, consistent with the linearly realized symmetries yield approximate functionals which are symmetric, when mean fields and Green's functions are transformed as $O(N)$-vectors and 2^{nd}-rank $O(N)$-tensors, respectively. However, the solutions of the dynamical equations of motion may imply a resummation to arbitrary orders in the considered expansion parameter[4]. Then this resummation is incomplete and crossing and $O(N)$-symmetry may be violated at orders beyond those explicitely included in the chosen approximant of Φ. This includes a violation of the β-function of the running coupling constant at the very same order (see also [17] and [II]).

The techniques developed in [I] can be used to renormalize the self-consistent self-energy and the 2PI Γ-functional with temperature-independent counter terms. As usual in the here considered case of a spontaneously broken symmetry the Γ-functional can be rendered finite at a mass scale $\tilde{\mu}^2 > 0$, i.e., in the symmetric Wigner-Weyl phase. Thus no particular problems with massless degrees of freedom appearing in the Nambu-Goldstone mode arise, while symmetries are violated at the self-consistent level.

The symmetry is appropriately repaired through the non-perturbative approximation (19) for the 1PI effective action functional. It was shown that all symmetries are recovered in the sense that the approximate proper vertex functions defined from (20) in the usual sense, i.e., by

$$\tilde{\Gamma}_{1,\ldots,n} = \mathrm{i}\frac{\delta^n\tilde{\Gamma}[\varphi]}{\delta\varphi_1\cdots\delta\varphi_n} \tag{42}$$

fulfill the usual WTIs of the proper vertex-functions and can be renormalized with temperature independent counter terms by the techniques developed in [I] and Sect. III C.

Based on the symmetry violating intermediate Dyson resummation, however, it is expected that certain relics of that *internal* stage are still present also in these symmetric proper vertex

[4] Exceptions are truncations right at zero order as all zero-order terms are self-replicative, see the $1/N$-expansion discussed in sect. V C

functions[5]. Since the propagators, which define the kernels of Bethe-Salpeter equations, are given by the symmetry violating self-consistent scheme, already the threshold structure is expected to deviate from the correct behavior, since the Nambu-Goldstone modes falsely appear with finite masses for these *internal* lines.

One could think to include the external propagator into the self-consistent scheme, i.e., to consider all quantities as a function of G_{ext} rather than the Dyson G. This could indeed further improve the approximation. However, since implicitly this defines a new Φ-derivable scheme, with a Φ defined by the 2PI diagrams leading to Σ_{ext}, it has the consequence that the symmetries may again be violated, although at a much more "remote" level.

IV. APPROXIMATIONS OF THE Φ-FUNCTIONAL

In this section we first give a functional derivation of the Φ-functional with the techniques developed in Sect. A up to order $\hbar^2$ to exemplify the reduction to 2PI diagrams and the symmetry properties of the approximations for the functional. As an application we give a diagrammatic derivation for the functional up to order λ^2.

We write the Lagrangian of the O(N)-model in the following form:

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \vec{\phi})(\partial^\mu \vec{\phi}) + \frac{\tilde{m}^2}{2}\vec{\phi}^2 - \frac{\lambda}{8}(\vec{\phi}^2)^2. \tag{43}$$

It is immediately clear that in this case for both, the loop expansion and the coupling constant expansion, the corresponding approximations for the Φ-functional are symmetric under O(N)-transformations, where φ and G transform as a vector and a tensor of 2[nd] rank, respectively.

A. The Φ-functional up to order $\hbar^2$

According to our discussion in Appendix A we have to calculate the functional

$$\begin{split} Z_1[J'] = & N \exp\left(\frac{\mathrm{i}}{\hbar}\tilde{S}[\varphi_0', J']\right) \int \mathrm{D}\Phi'' \exp\left\{\frac{\mathrm{i}}{2}\int_{\mathcal{C}} \mathrm{d}(12)\ (\mathscr{G}^{-1})_{1j,2k} (\phi'')_1^j (\phi'')_2^k \right. \\ & \left. - \frac{\mathrm{i}\lambda}{24}\delta_{jklm}\int_{\mathcal{C}}\mathrm{d}(1)\ \left(\hbar (\phi'')_1^j(\phi'')_1^k(\phi'')_1^l(\phi'')_1^m + 4\sqrt{\hbar}(\phi'')_1^j(\phi'')_1^k(\phi'')_1^l\varphi_1^m\right)\right\}, \\ & \text{where} \quad \delta_{jklm} = \delta_{jk}\delta_{lm} + \delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl} \end{split} \tag{44}$$

and where we have explicitly reintroduced the internal field indices. The $\hbar$-expansion can now be generated following the standard technique for the usual perturbative calculation (see, e.g., [18]). All path integrals of the $\hbar$ expansion can be found be taking functional derivatives of the Gaussian path integral

$$\begin{split} Z_{10}[J', K] &= N \int \mathrm{D}\Phi'' \exp\left(\int_{\mathcal{C}} \mathrm{d}(12)\ (\mathscr{G}^{-1})_{1j,2k}(\phi'')_1^j(\phi'')_2^k\right) \\ &= \exp\left[-\frac{1}{2}\operatorname{Tr}\ln(\mathscr{G}^{-1}/M^2) - \frac{\mathrm{i}}{2}\int_{\mathcal{C}}\mathrm{d}(12)\ \mathscr{G}_{12}^{jk} K_{1j} K_{2k}\right] \end{split} \tag{45}$$

---

[5] This is essentially true for all approximations other then self-consistent ones, e.g., in perturbation theory the internal structure is given by free particle properties!

with respect to the new independent local auxiliary source K which has to be set to 0 at the end of the calculation. We have to substitute only

$$(\phi'')_1^j \rightarrow \frac{1}{\mathrm{i}} \frac{\delta}{\delta K_{1j}} \tag{46}$$

in the polynomial expression for the expansion of $\exp(S_I[\sqrt{\hbar}\phi'', \varphi])$.

In our case this leads to

$$\begin{aligned} W_1[J'] &= -\mathrm{i}\hbar \ln[Z_{10}[J', K=0](1 + \hbar z_1^{[1]} + \hbar^2 z_1^{[2]})] + O(\hbar^3) \\ &= \tilde{S}[\varphi_0', J'] + \frac{\mathrm{i}\hbar}{2} \operatorname{Tr}\ln(\mathscr{G}^{-1} M^2) \\ &\quad + \hbar^2 \left(\frac{\lambda}{8} \delta_{jklm} \int_{\mathcal{C}} \mathrm{d}(1)\; \mathscr{G}_{11}^{jk} \mathscr{G}_{11}^{lm} + \frac{\lambda^2}{8} \delta_{jklm} \delta_{j'k'l'm'} \int_{\mathcal{C}} \mathrm{d}(12)\; \varphi_1^j \mathscr{G}_{11}^{kl} \mathscr{G}_{12}^{mm'} \mathscr{G}_{22}^{k'l'} \varphi_2^{j'} \} \right). \end{aligned} \tag{47}$$

Now we need to substitute $\varphi' = 0$ instead of φ_0' in this expression. To that end we have to find the expression for

$$\varphi' = \frac{\delta W_1[J']}{\delta J'} \tag{48}$$

up to order $\hbar$. Since φ_0 is the stationary point of $\tilde{S}[\phi', J']$ at fixed J' we have

$$\frac{\delta \tilde{S}[\varphi_0', J']}{\delta J'} = \varphi_0'. \tag{49}$$

Further from the equation of motion for φ_0' we have

$$\frac{\delta (\varphi_0')_1^j}{\delta (J')_{2k}} = \left[\left(\frac{\delta J'}{\delta \varphi_0'} \right)^{-1} \right]_{12}^{jk} = \mathscr{G}_{12}^{jk}. \tag{50}$$

From this we get

$$\frac{\mathrm{i}\hbar}{2} \frac{\delta}{\delta (J')_1^j} \operatorname{Tr}\ln(\mathscr{G}^{-1}/M^2) = \frac{\mathrm{i}\hbar}{2} \lambda \delta_{klmn} \int_{\mathcal{C}} \mathrm{d}(1')\; G_{1'1'}^{kl} \varphi_{1'}^m G_{1'1}^{nj} + O(\hbar^2) \tag{51}$$

and thus

$$\begin{aligned} \frac{\mathrm{i}\hbar}{2} \operatorname{Tr}\ln(\mathscr{G}^{-1}/M^2) =& \frac{\mathrm{i}\hbar}{2} \operatorname{Tr}\ln(G^{-1} M^2) \\ & - \frac{\hbar^2}{4} \lambda^2 \delta_{jklm} \delta_{j'k'l'm'} \int_{\mathcal{C}} \mathrm{d}(12)\; \varphi_1^j G_{11}^{kl} G_{12}^{mm'} G_{22}^{l'k'} \varphi_2^{j'} + O(\hbar^3) \end{aligned} \tag{52}$$

and

$$\tilde{S}[\varphi_0'] = \frac{\hbar^2}{8} \lambda^2 \delta_{jklm} \delta_{j'k'l'm'} \int_{\mathcal{C}} \mathrm{d}(12)\; \varphi_1^j G_{11}^{kl} G_{12}^{mm'} G_{22}^{l'k'} \varphi_2^{j'} + O(\hbar^3) \tag{53}$$

Gathering all terms finally yields

$$\begin{aligned} \Gamma[\varphi, G] &= S[\varphi] - \frac{\mathrm{i}\hbar}{2} \int_{\mathcal{C}} \mathrm{d}(12)\; \mathscr{D}_{12}^{-1} (G_{12} - \mathscr{D}_{12}) + W_1[J_0'] \\ &= S[\varphi] - \frac{\mathrm{i}\hbar}{2} \int_{\mathcal{C}} \mathrm{d}(12)\; \mathscr{D}_{12}^{-1} (G_{12} - \mathscr{D}_{12}) \\ &\quad + \frac{\mathrm{i}\hbar}{2} \operatorname{Tr}\ln(G_{12}^{-1}/M^2) + \frac{\hbar^2 \lambda}{8} \delta_{jklm} \int_{\mathcal{C}} \mathrm{d}(1)\; G_{11}^{jk} G_{11}^{lm} \\ &\quad\quad + \frac{\hbar^2 \lambda^2}{12} \delta_{jklm} \delta_{j'k'l'm'} \int_{\mathcal{C}} \mathrm{d}(12)\; \varphi_1^j G_{12}^{kk'} G_{12}^{ll'} G_{12}^{mm'} \varphi_2^{j'} + O(\hbar^3). \end{aligned} \tag{54}$$

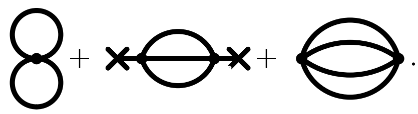
This is indeed invariant under global $\mathrm{O}(N)$ operations if the mean fields φ and G are transformed as tensors of 1st and 2nd rank respectively.

B. The Φ-functional to order λ^2

Now we like to give an example for the application of the diagram rules for the Φ-functional and the equations of motion for the self-consistent field and self-energy.

From (A10) we see that the Φ-functional is given by diagrams which obey the Feynman rules with the point vertices of a field theory with an interaction Lagrangian given by $S_I[\phi, \varphi]$. The Lines, connecting these point vertices denote full propagators iG rather than perturbative propagators. As argued in the paragraph after equation (A17) one has to keep only two-particle irreducible closed diagrams with at least two loops for iΦ.

For the $\mathrm{O}(N)$-model this leads to the following diagram expression for iΦ up to order λ^2 in the coupling-constant expansion scheme

$$\mathrm{i}\Phi[\varphi, G] =$$

(55)

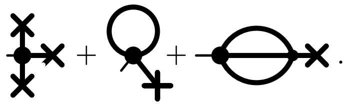
The corresponding analytic expression reads

$$\begin{aligned}\Phi[\varphi, G] =& \frac{\hbar^2\lambda}{8}\delta_{jklm}\int_{\mathcal{C}} \mathrm{d}(1)\; G_{11}^{jk} G_{11}^{lm} \\ &+ \frac{\hbar^2\lambda^2}{12}\delta_{jklm}\delta_{j'k'l'm'}\int_{\mathcal{C}} \mathrm{d}(12)\; \varphi_1^j G_{12}^{kk'} G_{12}^{ll'} G_{12}^{mm'} \varphi_2^{j'} \\ &+ \frac{\mathrm{i}\hbar^3\lambda^2}{48}\delta_{jklm}\delta_{j'k'l'm'}\int_{\mathcal{C}} \mathrm{d}(12)\; G_{12}^{jj'} G_{12}^{kk'} G_{12}^{ll'} G_{12}^{mm'} + O(\lambda^3).\end{aligned} \tag{56}$$

The equations of motion for the approximation are given by (A16-A17). For the mean field we find

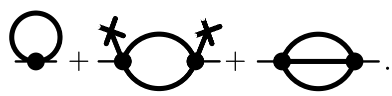
$$\begin{aligned}\mathrm{i}(-\Box + \tilde{m}^2)\varphi_1^j - \frac{\mathrm{i}\lambda}{6}\delta_{jklm}\varphi_1^k\varphi_1^l\varphi_1^m + \frac{\hbar\lambda}{2}\delta_{j'k'jm}\varphi_1^m G_{11}^{j'k'} \\ + \frac{\mathrm{i}\hbar^2\lambda^2}{6}\delta_{jklm}\delta_{j'k'l'm'}\int_{\mathcal{C}} \mathrm{d}(2)\; G_{12}^{kk'} G_{12}^{ll'} G_{12}^{mm'} \varphi_2^{j'} \stackrel{!}{=} 0.\end{aligned} \tag{57}$$

In terms of diagrams this equation of motion reads

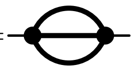
$$-\mathrm{i}(-\Box + \tilde{m}^2)\varphi_1 =$$

(58)

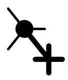
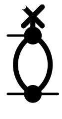
The equation for the self-consistent self-energy follows from (4)

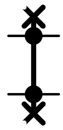
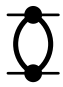
$$\begin{aligned}-\mathrm{i}\Sigma_{1j,2k} =& \frac{\mathrm{i}\hbar\lambda}{2}\delta_{jklm} G_{11}^{lm}\delta_{12}^{(d)} + \frac{\mathrm{i}\hbar\lambda^2}{2}\delta_{jpqr}\delta_{kp'q'r'}\varphi_1^p G_{12}^{rr'} G_{12}^{qq'}\varphi_2^{p'} \\ &- \frac{\hbar^2\lambda^2}{6}\delta_{jpqr}\delta_{kp'q'r'} G_{12}^{pp'} G_{12}^{qq'} G_{12}^{rr'}\end{aligned} \tag{59}$$

$$=$$


According to (31-34) the diagrammatical elements for the calculation of the external self-energy are given by

$$\frac{\partial \mathrm{i}\Phi[\varphi, G]}{\delta\varphi_1\delta\varphi_2} =$$  , (60)

$$\mathrm{i}\Gamma^{(3)} =$$  +  , (61)

$$\mathrm{i}\Gamma^{(4)} =$$  +  +  . (62)

V. A SIMPLE EXAMPLE: THE HARTREE (PLUS EXCHANGE) APPROXIMATION

To illustrate the above formal considerations in this section we numerically solve the equations of motion for both the self-consistent and the symmetric effective self-energies for the approximation keeping only terms to linear order in the explicitely appearing coupling λ in all above equations. This defines the Hartree approximation including the corresponding bosonic exchange terms[6]. Especially Φ, c.f. (56), becomes

$$\Phi[\varphi, G] = \frac{\lambda}{8}\delta_{jklm}\int_{\mathcal{C}} \mathrm{d}(1)\; G_{11}^{jk} G_{11}^{lm}, \tag{63}$$

where we have set $\hbar = 1$.

A. The vacuum case

In the following we can restrict ourselves to time ordered functions, so that in this section all propagators stand for $\{--\}$-propagators and all vertices for $\{-\}$-vertices.

First we have to find and solve the self-consistent equations of motion (3) for the approximation given by the Φ functional (63). For a given mean field φ_1^j we find:

$$(\mathscr{D}^{-1})_{1j,2k} = \left[(-\Box_1 + \tilde{m}^2)\delta_{jk} - \frac{\lambda}{2}(2\varphi_{1j}\varphi_{1k} + \vec{\varphi}_1^2\delta_{jk})\right]\delta^{(d)}(x_1 - x_2). \tag{64}$$

Since the vacuum is homogeneous in space and time the mean field is constant and pointing in one arbitrarily chosen direction. Therefore it is convenient to express all vector and tensor quantities in terms of their components parallel and perpendicular to $\vec{\varphi}$. The perpendicular modes are those of the Goldstone bosons (e.g., the pions in the linear sigma model). The corresponding projectors are

$$\begin{aligned} P_{\perp}^{jk} &= \delta^{jk} - \frac{\varphi^j\varphi^k}{\vec{\varphi}^2}, \\ P_{\parallel}^{jk} &= \frac{\varphi^j\varphi^k}{\vec{\varphi}^2}. \end{aligned} \tag{65}$$

[6] The latter are of next-to-leading order in the semi-classical $1/N$ expansion.

In momentum representation the self-energy components derived from the approximation to Φ given by (63) become

$$\begin{aligned} \Sigma_{\perp} &= \frac{\mathrm{i}\lambda}{2}\mu^{2\epsilon} \int \frac{\mathrm{d}^d l}{(2\pi)^d} [(N+1)G_{\perp}(l) + G_{\parallel}(l)], \\ \Sigma_{\parallel} &= \frac{\mathrm{i}\lambda}{2}\mu^{2\epsilon} \int \frac{\mathrm{d}^d l}{(2\pi)^d} [(N-1)G_{\perp}(l) + 3G_{\parallel}(l)]. \end{aligned} \tag{66}$$

Correspondingly the Dyson equations (17) decouple with

$$\begin{aligned} G_{\perp}(p) &= \frac{1}{p^2 - M_{\perp}^2 + \mathrm{i}\eta}, \\ G_{\parallel}(p) &= \frac{1}{p^2 - M_{\parallel}^2 + \mathrm{i}\eta} \\ \text{with } M_{\perp}^2 &= \frac{\lambda}{2}\vec{\varphi}^2 - \tilde{m}^2 + \Sigma_{\perp}, \\ M_{\parallel}^2 &= \frac{3\lambda}{2}\vec{\varphi}^2 - \tilde{m}^2 + \Sigma_{\parallel}. \end{aligned} \tag{67}$$

With these definitions the equation of motion for the mean field reads

$$\varphi_l(M_{\parallel}^2 - \lambda\vec{\varphi}^2) = 0, \tag{68}$$

which shows that either the mean field or the bracket vanishes. In the first case, the symmetric Wigner-Weyl mode, in the second the Nambu-Goldstone mode is realized. In our case of negative $-\tilde{m}^2$ the spontaneously broken phase is realized in the vacuum.

In the following we shall use the convention and renormalization prescription according to (40). In this renormalization scheme the self-consistent gap equations (66-67) become

$$\begin{aligned} M_{\perp}^2 = \Theta_{\perp}(M_{\perp}, M_{\parallel}) &:= \frac{\lambda}{2}\vec{\varphi}^2 - \tilde{m}^2 + \frac{\lambda}{32\pi^2}\Bigg[(N+2)\tilde{\mu}^2 - (N+1)M_{\perp}^2 - M_{\parallel}^2 \\ &\qquad + M_{\perp}^2(N+1)\ln\left(\frac{M_{\perp}^2}{\tilde{\mu}^2}\right) + M_{\parallel}^2\ln\left(\frac{M_{\parallel}^2}{\tilde{\mu}^2}\right)\Bigg] \\ M_{\parallel}^2 = \Theta_{\parallel}(M_{\perp}, M_{\parallel}) &:= \frac{3\lambda}{2}\vec{\varphi}^2 - \tilde{m}^2 + \frac{\lambda}{32\pi^2}\Bigg[(N+2)\tilde{\mu}^2 - (N-1)M_{\perp}^2 - 3M_{\parallel}^2 \\ &\qquad + M_{\perp}^2(N-1)\ln\left(\frac{M_{\perp}^2}{\tilde{\mu}^2}\right) + 3M_{\parallel}^2\ln\left(\frac{M_{\parallel}^2}{\tilde{\mu}^2}\right)\Bigg] \end{aligned} \tag{69}$$

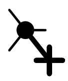
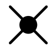
The equations show clearly that the physical results are independent of the choice of the mass renormalization scale $\tilde{\mu}$ since a different choice of $\tilde{\mu}$ can be compensated by a (finite) renormalization of the coupling λ and the mass parameter $\tilde{m}$. For more general Φ-derivable approximations, which contain real two-point contributions to the self-energy, also a wave function renormalization is needed (see [6]). The introduction of the mass renormalization scale $\tilde{\mu}$ is necessary only because we need a "*mass independent renormalization scheme*" [19, 20] which uses the fact that a theory with symmetries can be renormalized with counter terms of its symmetric realization in the Wigner-Weyl mode also for the spontaneously broken Nambu-Goldstone phase. In addition mass independent renormalization schemes have the advantage to avoid "renormalization induced" infrared divergences which would appear with the massless modes within an on-shell renormalization scheme. It is also clear that for special choices of $\tilde{\mu}$ our class of renormalization schemes contains the minimal subtraction scheme (MS) and the modified minimal subtraction scheme ($\overline{\text{MS}}$).

From (69) it becomes immediately clear that by the self-consistent Φ-derivable approximation the symmetry is explicitly broken in the Nambu-Goldstone mode of the theory, i.e., for $m^2 = -\tilde{m}^2 < 0$ which provides $\varphi^2 > 0$: The transverse modes should be massless according to Goldstone's theorem which does not hold true according to (69).

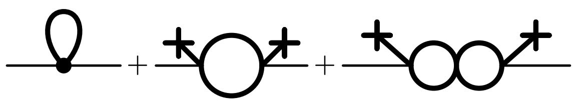
As we have seen in section III we can define an approximation to the self-energy respecting the underlying symmetry by (25). For that purpose we need to solve the equation of motion (30). From (31) we obtain the kernels, here given in the momentum representation

$$\begin{aligned} \Gamma^{(4)}_{jk;l,m} &= -\frac{\lambda}{2}(\delta_{jk}\delta_{lm} + \delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}) := -\frac{\lambda}{2}\delta_{jklm}, \\ \Gamma^{(3)}_{jk;l} &= -\lambda(\varphi_k\delta_{jl} + \varphi_j\delta_{kl} + \varphi_l\delta_{jk}) = -\lambda\delta_{jklm}\varphi_m. \end{aligned} \tag{70}$$

Since due to translation invariance φ is independent of the four-momentum both $\Gamma^{(4)}$ and $\Gamma^{(3)}$ are constant and effectively only one-point functions. From this it is clear that $\Lambda^{(3)}$ is effectively only a two-point function and thus depends only on the momentum attached to its third argument. This becomes also immediately clear from the diagrams (32) and (34). In our special case the kernels $\Gamma^{(3)}$ and $\Gamma^{(4)}$ are point vertices

$\mathrm{i}\Gamma^{(3)} =$  and $\mathrm{i}\Gamma^{(4)} =$  (71)

and thus from the iterative solution of the equation of motion (30) and (33) for $\Lambda^{(3)}$ we conclude that the external self-energy associated with the Hartree approximation is given by the *RPA bubble resummation*

$-\mathrm{i}\Sigma_{\text{ext}} =$  $+ \cdots$ (72)

To find the analytic expressions for this diagrammatic equation we specialize (30) with the kernels (70). This leads to the equation

$$\Lambda^{(3)}_{jkl}(k) = -\lambda\delta_{jklm}\varphi^m + \frac{\mathrm{i}\lambda}{2}\delta_{jkl'm'}\Lambda^{(3)}_{l''m'';l}(k)\int\frac{\mathrm{d}^d l}{(2\pi)^d}G^{l'l''}(l)G^{m'm''}(l-k). \tag{73}$$

According to (40) for renormalization we have to subtract the expression where in (73) both propagators are set to free propagators with the mass parameter set to the mass renormalization scale $\tilde{\mu}$ and $k = 0$.

From the symmetry properties of $\Lambda^{(3)}$ it follows that it must be of the form

$$\begin{aligned} \Lambda^{(3)}_{jk;l}(k) =& \varphi^l\left(\Lambda^{(3)}_1(k)P^{jk}_{\perp} + \Lambda^{(3)}_2(k)P^{jk}_{\parallel}\right) \\ &+ \Lambda^{(3)}_3(k)\left(\varphi^j P^{kl}_{\perp} + \varphi^k P^{jl}_{\perp}\right) \\ &+ \Lambda^{(3)}_4(k)\left(\varphi^j P^{kl}_{\parallel} + \varphi^k P^{jl}_{\parallel}\right). \end{aligned} \tag{74}$$

Indeed after some algebraic manipulations with the propagators (67) we obtain the algebraic linear equations of motion for the four independent components of Λ^3 as

$$\begin{aligned}
\Lambda_1^{(3)}(k) &= -\lambda + \frac{\lambda}{2}\left[(N+1)L_{\perp,\perp}(k)\Lambda_1^{(3)}(k) + (\Lambda_2^{(3)}(k) + 2\Lambda_4^{(3)}(k))L_{\|,\|}(k)\right], \\
\Lambda_2^{(3)}(k) &= -\lambda + \frac{\lambda}{2}\left[(N-1)L_{\perp,\perp}(k)\Lambda_1^{(3)}(k) + (3\Lambda_2^{(3)}(k) + 2\Lambda_4^{(3)}(k))L_{\|,\|}(k)\right], \\
\Lambda_3^{(3)}(k) &= -\lambda + \lambda L_{\|,\perp}(k)\Lambda_3^{(3)}(k), \\
\Lambda_4^{(4)}(k) &= -\lambda + \lambda L_{\|,\|}(k)\Lambda_4^{(3)}(k).
\end{aligned} \tag{75}$$

Here we used the abbreviation

$$\begin{gathered}
L_{\alpha,\beta}(k) = L_{\beta,\alpha}(k) = \mathrm{i}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\left\{G_\alpha(l)G_\beta(l-k) - [D(l)]^2\right\} \\
\text{with } \alpha,\beta \in \{\perp,\|\} \text{ and } D(l) = \frac{1}{l^2 - \tilde{\mu}^2 + \mathrm{i}\eta}.
\end{gathered} \tag{76}$$

According to (33) and (73) the external self-energy is given by

$$\begin{aligned}
(\Sigma_{\mathrm{ext}})_{jk}(p) &= \Sigma_{jk}(p) - \varphi^l(\Lambda_{lj;k}^{(3)}(p) - \Gamma_{ljk}^{(3)}) \\
&= \Sigma_{jk} - \vec{\varphi}^2\{[3\lambda + \Lambda_2^{(3)}(p) + 2\Lambda_4^{(3)}(p)]P_{\|jk} + [\lambda + \Lambda_3^{(3)}(p)]P_{\perp jk}\}.
\end{aligned} \tag{77}$$

In order to prove Goldstone's theorem we have to set $p = 0$ and use the equations of motion (68) and (69). The crucial property is

$$M_\perp^2 = \Sigma_\perp - \Sigma_\| = -\lambda[\Theta_1(M_\|^2) - \Theta_1(M_\perp^2) - (M_\|^2 - M_\perp^2)\Theta_2(\tilde{\mu}^2)], \tag{78}$$

where the functions Θ_1 and Θ_2 are given in appendix C cf. (C1) and (C2). Using (C5) together with

$$M_{\mathrm{ext}\perp}^2 = \frac{\lambda}{2}\vec{\varphi}^2 - \tilde{m}^2 + \Sigma_{\mathrm{ext}\perp}(0) \tag{79}$$

and (77) we find indeed $M_{\mathrm{ext}\perp}^2 = 0$ as it should be according to Goldstone's theorem: The $(N-1)$ degrees of freedom perpendicular to the mean field are the Goldstone modes corresponding to the spontaneous breaking of the $\mathrm{O}(N)$ to the $\mathrm{O}(N-1)$ symmetry group of the mean field.

Finally we remark that the Hartree self-energy for the case of the Wigner-Weyl phase of the theory, where $\vec{\varphi} = 0$, is identical with the external self-energy and thus fulfills also the WTI for the self-energy, which indeed in this case is trivially fulfilled.

In Fig. 1 we plot an example for the above considerations: We roughly fit the parameters of the self-consistent self-energy such that the properties for pions are satisfied, namely $M_\perp = 140\mathrm{MeV}$ and $M_\| = 600\mathrm{MeV}$, $\varphi = f_\pi = 93\mathrm{MeV}$, and $n = 4$. The equations of motion (69) were solved for $\tilde{m}^2$ and $\tilde{\mu}^2$. The renormalization conditions (40) were used. The plot of the pion mass clearly shows that Goldstone's theorem is recovered, since $(M_\perp^{(\mathrm{ext})})^2(s = 0) = 0$ although the internal Green's functions do not fulfill the Ward identities. This violation of the symmetry properties by the self-consistent self-energies is clearly seen in the spectral function for the "σ-meson", since its threshold is at $\sqrt{s} = 0 = 2M_\perp = 280\mathrm{MeV}$ and not at $\sqrt{s} = 0$ as it should be for massless pions in the chiral limit.

Like the here constructed *external* self-energy where due to the finite mean field φ the RPA-bubbles contributed to the self-energy expectation value $\varphi(x)\varphi(y)\left\langle\phi^2(x)\phi^2(y)\right\rangle$ also other symmetry preserving two-point functions are given by the very same RPA terms, and thus result from the linear response of the system due to fluctuations around the Hartree solution. A prominent example is the correlator $\langle j^\mu(x)j^\nu(y)\rangle$ of the Noether current (11), which then is conserved.

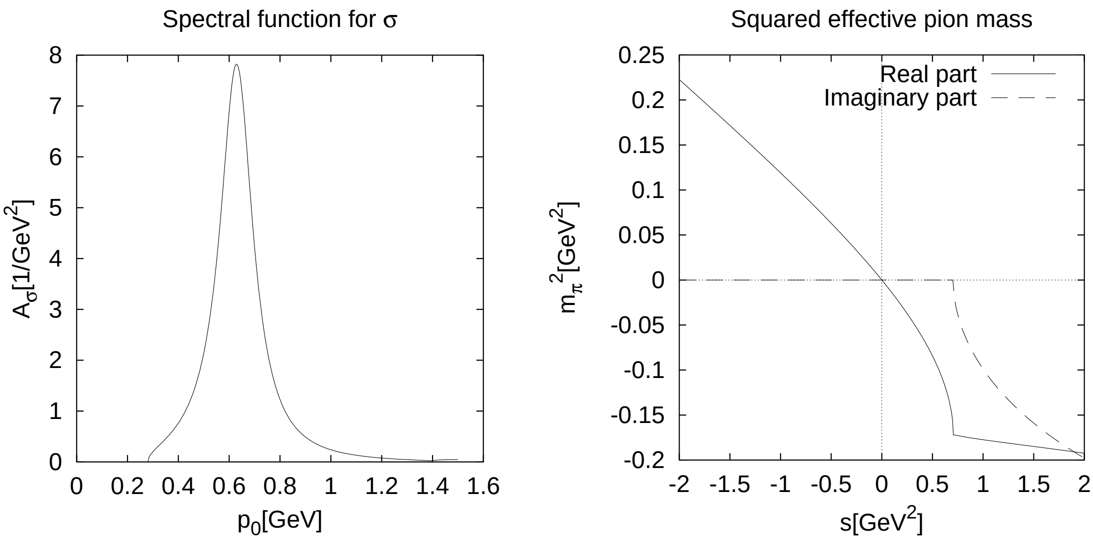


FIG. 1: The "σ" spectral function (left) and the external effective mass of the "pions" (right).

B. The finite temperature case

Since the counter terms at finite temperature are the same as for the vacuum case we can immediately write down the renormalized equations of motion where we make use of the vacuum functions defined in (69) and the explicitly T-dependent finite part of the tadpole diagram:

$$\Theta^{(T)}(M) = \frac{\mathrm{i}\lambda}{2} \int \frac{\mathrm{d}^4 l}{(2\pi)^4} 2\pi n_T(l_0)\delta(l^2 - M^2) = \frac{\mathrm{i}\lambda}{4\pi^2} \int_M^\infty \mathrm{d}\omega \sqrt{\omega^2 - M^2} n_T(\omega), \tag{80}$$

with the Bose-Einstein distribution function n_T.

From the fact that the self-consistent $\{--\}$-propagator reads

$$\begin{aligned} G^{--}(p, M) &= \frac{1}{p^2 - M^2 + \mathrm{i}\eta} - 2\pi \mathrm{i} n(p_0)\delta_\eta(p^2 - M^2), \\ \delta_\eta(x) &= \frac{1}{2\pi \mathrm{i}} \operatorname{Im}\left(\frac{1}{x - \mathrm{i}\eta} - \frac{1}{x + \mathrm{i}\eta}\right) \end{aligned} \tag{81}$$

and that for our tadpole integral we are allowed take the limit $\eta \to +0$ in the explicitly temperature dependent part we find the renormalized gap equation at finite temperature

$$\begin{aligned} 0 &= \varphi(M_\parallel^2 - \lambda\varphi^2), \\ M_\perp^2 &= \Theta_\perp(M_\perp, M_\parallel) + (N+1)\Theta^{(T)}(M_\perp) + \Theta^{(T)}(M_\parallel), \\ M_\parallel^2 &= \Theta_\parallel(M_\perp, M_\parallel) + (N-1)\Theta^{(T)}(M_\perp) + 3\Theta^{(T)}(M_\parallel), \end{aligned} \tag{82}$$

where $\Theta_\perp$ and $\Theta_\parallel$ are defined by (69)[7]. The solutions of these gap-equations are shown in Fig. 2 which shows clearly a first order phase transition behavior. There exist two "critical temperatures": For $0 < T < T_{c1}$ there exists only one solution with $\varphi \neq 0$, for $T_{c1} < T < T_{c2}$ two solutions with $\varphi \neq 0$ and the symmetric solution with $\varphi = 0$ while for $T > T_{c2}$ the only solution is symmetric. Also

[7] Note that although the self-consistent masses $M_\perp$ and $M_\parallel$ are temperature dependent the counter terms used to render $\Theta_\perp$ and $\Theta_\parallel$ finite are temperature-independent since the counter terms $\propto \Sigma_\perp$ and $\Sigma_\parallel$ are due to the subtraction of hidden vacuum-divergences of the four-point function as explained in detail in [I].

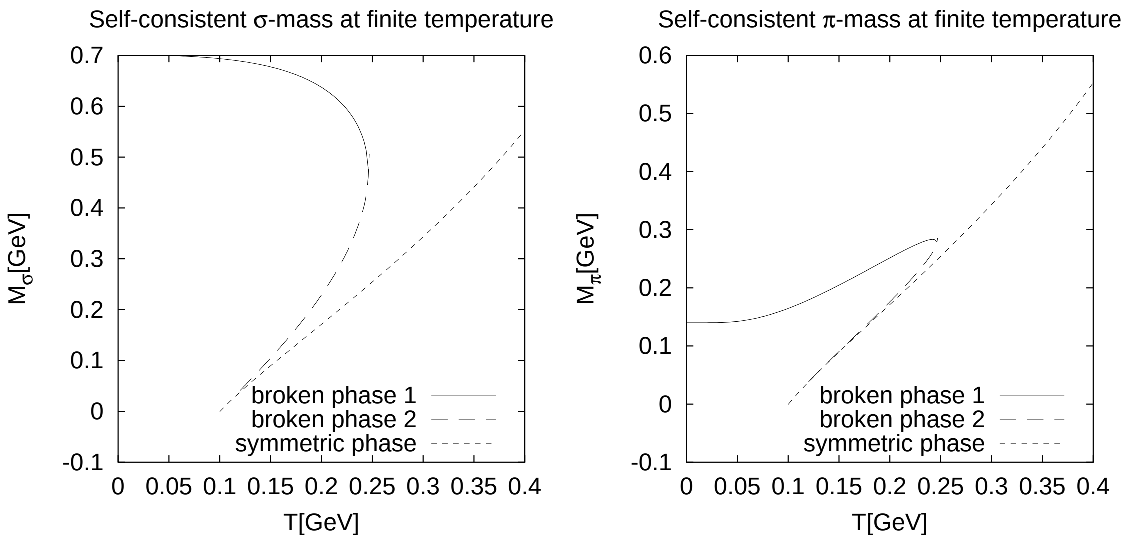


FIG. 2: The solutions for the gap equations (82).

the equations of motion for the external self-energy (77) remain formally the same but has to be read within the real-time $\{-+\}$-matrix formalism. As cited in [I] appendix A the analytic properties of two-point functions lead to the conclusion that the retarded (and also the advanced) propagators and self-energies decouple from the other degrees of freedom, so that the simple algebraic properties are valid for them as in the vacuum. We make use of the property

$$F^R(p) = \operatorname{Re} F^{--}(p) + \mathrm{i} \tanh\left(\frac{p_0\beta}{2}\right) \operatorname{Im} F^{--}(p), \tag{83}$$

which holds true for any amputated two-point function.

In our special case we can take advantage of these analytic properties also for the functions $\Lambda^{(4)}$, since effectively these are two-point functions as well. Thus we can use the vacuum equations (75) for the retarded functions without changes. Furthermore it is clear that these functions do not contain any renormalization parts except the already removed pure vacuum divergences.

Since the effective mass at $p = 0$ is identical with the second derivative of the effective potential, defined by

$$V_{\text{eff}}[\varphi]\delta^{(4)}(p) = -\tilde{\Gamma}[\varphi]|_{\varphi=\text{const.}} \tag{84}$$

its value should be ≥ 0 for a stable solution, i.e., for a minimum of the effective potential rather than a maximum which provides an unstable "tachyonic" solution. The explicit calculation shows that the solution, denoted by "broken phase 2" in Fig. 2 are unstable. This shows that we find a phase transition of first order, i.e., a discontinuity in the order parameter $\varphi^2 = M_{\|}^2/\lambda$ (see Fig. 1).

The effective masses for the stable spontaneously broken phase together with the spectral function for the σ-meson are depicted in Fig. 3.

The calculation clearly shows the symmetry violations of the underlying self-consistent Dyson approximation: Although the WTIs and Goldstone's theorem are fulfilled for the external propagator remnants of their violation by the internal propagators are present: The low-energy threshold behavior of the "σ-meson" is not correct since the "pion" mass within the internal propagator does not vanish. Also the phase transition comes out to be of first instead of second order as it should be.

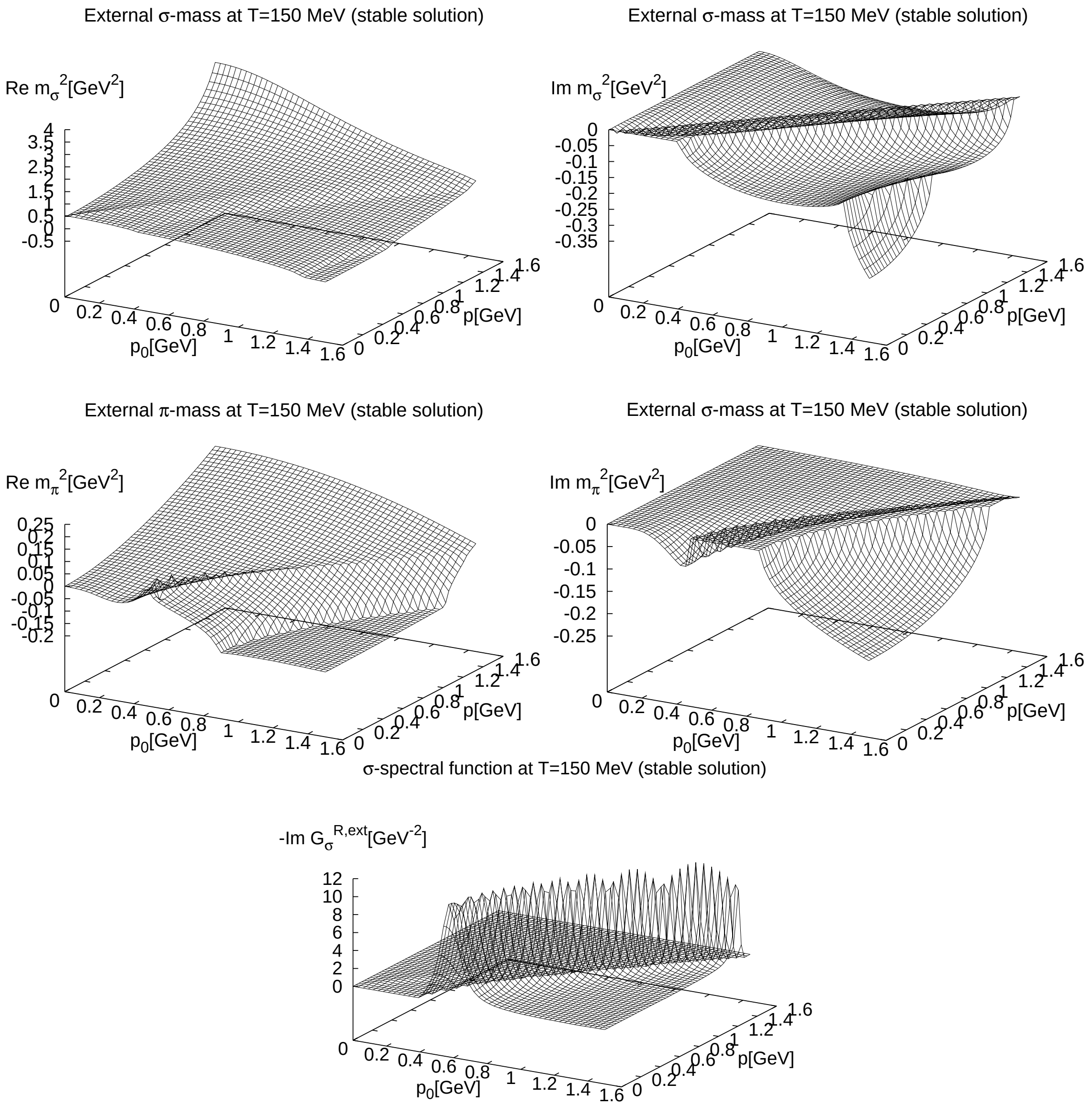


FIG. 3: The effective external masses at a temperature of 150MeV. The effective external π-mass indeed vanishes at $p_0 = \vec{p} = 0$ as predicted from Goldstone's theorem. The spectral function of the σ-meson shows that at high temperatures its strength becomes more peaked and the maximum shifted to lower momenta than at $T = 0$.

C. Leading order large-N

In the context of symmetries the large N expansion scheme is of particular importance. Here N denotes the number of fields, e.g., in SU(N) or O(N) theories. The counting depends on the type of theory and is defined such that the classical action scales like N, c.f. [14, 21]. As unrestricted loops scale like N this implies that coupling and mean fields scale like

$$\text{unrestricted loops} \propto N, \quad \lambda \propto \frac{1}{N}, \quad \vec{\varphi} \propto \sqrt{N} \tag{85}$$

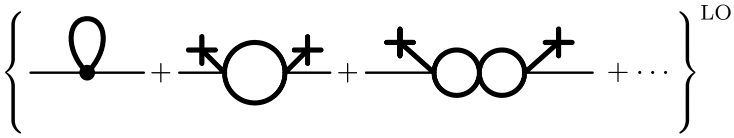
for the here considered O(N) model in the Nambu-Goldstone phase. As the WTIs concern the self-energies in the first place we shall first dicuss the $1/N$ expansion at this level before we comment on it in the context of the 2PI-functional formalism.

$1/N$-expansion of the self-energy

The $1/N$ counting scheme has the remarkable feature that the leading order (LO) leads to zero order terms, i.e., $\propto (1/N)^0$, in the self-energies. This implies that iterative self-energy insertions of zero order contribute in LO which amends the entire LO self-energy self-consistently to be constructed within a corresponding Dyson resummation. Thus the resulting LO-propagator is created fully self-consistently and at the same time accounts for *all* terms at LO. The latter fact guarantees that for the LO self-energies the symmetries, i.e., the corresponding WTIs, are fulfilled.

Since the above derived Hartree + RPA (external) self-energy, given by Eqs. (66) to (77) indeed includes all zero order terms in $1/N$-expansion, the leading order can simply by obtained by retaining the according large N limit terms

$$-\mathrm{i}\Sigma^{\mathrm{LO}} = \quad . \qquad (86)$$



Naturally this leads to counting factors different from those in the RPA result (72) as different contractions of the $(\vec{\varphi}\vec{\varphi})^2$ interaction term lead to different $1/N$-orders (see, e.g., [14]). The counting further implies that (a) solely loops in the Nambu-Goldstone modes survive and (b) therefore the bubble sum only contributes to the σ-meson self-energy $\Sigma_{\|}$. The explicit result is

$$\begin{aligned} \Sigma_{\perp}^{\mathrm{LO}} &= \frac{\mathrm{i}N\lambda\mu^{2\epsilon}}{2}\int\frac{\mathrm{d}^d l}{(2\pi)^d} G_{\perp}^{\mathrm{LO}}(l), \qquad \text{with } (M_{\perp}^{\mathrm{LO}})^2\varphi = 0, \\ \Sigma_{\|}^{\mathrm{LO}} &= \Sigma_{\perp}^{\mathrm{LO}} + \frac{\varphi^2\lambda^2 N L_{\perp,\perp}^{\mathrm{LO}}}{1-\lambda N L_{\perp,\perp}^{\mathrm{LO}}}. \end{aligned} \qquad (87)$$

Here $L_{\perp,\perp}^{\mathrm{LO}}$ denotes the pion bubble loop, c.f. (76). The result clearly shows that for the broken phase $\varphi_l \neq 0$ the "pion"-mass vanishes for the zero-order propagator. The renormalization of Eq. (87) is done in the same way as described above for the Hartree case and of course also the renormalized "pion"-mass vanishes for both, the vacuum and the finite-temperature case. The WTI for the self-energy (87) are fulfilled by construction, if all terms of a given order are included, and thus both the O(N)-Noether current (11) is conserved and the Goldstone theorem is fulfilled. For the zero-order result (87) all this can be proven by mere inspection.

Please note that although the LO result (87) can be generated from a self-consistent Dyson resummation scheme, it is *not* Φ*-derivable*, since the pion self energy pieces corresponding to the bubble series are of subleading order! Higher order approximations can even not be expressed in terms of a self-consistent scheme, since from the Dyson series the corresponding propagator contains all orders, while the terms contributing to the self energy are of limited order!

$1/N$-expansion of the 2PI generating functional

In recent times the $1/N$-expansion scheme also has been used to organize truncation schemes for the here considered 2PI generating functional. Thereby the counting results from the scaling rules (85), however excluding the internal structure of the self-consistent propagators from the counting in the diagrams of Φ (1), c.f. refs. [14, 21]. This procedure leads to a 2PI generating function which is symmetric, a premise for the entire discussion in the papers.

In leading order $1/N$ one recovers the diagrams for the Hartree approximation (64) sect. V and the corresponding Dyson resummation result, however with the counting factor arising from

LO in $1/N$. The internal self-energy resulting from the corresponding Dyson resummation is given by the above result (86), (87), however with the essential difference that the bubble contributions are dropped. The reason is that the corresponding 2PI diagrams, c.f. Fig. 5 in ref. [14], are of subleading order. Nevertheless the pion mass is still zero, since it is not affected from this difference, while the sigma-meson self-energy is lacking the corresponding decay cuts and the WTIs at the correlator level are indeed violated. The external self-energy then exactly recovers the missing diagrams leading back to the result (87). In fact for the 2PI-$1/N$-expansion the Goldstone theorem can only be assured for the so called mass-matrix which is identical to the here considered external self-energy. A noteworthy side feature, though, is that the LO approximation gives the correct 2^{nd}-order phase transition. For details see [10].

As the main theme of this paper we explicitely see here that a symmetric generating functional by itself does not guarantee that the solution of the equations of motion, i.e., the Dyson equation, preserves the symmetry. Rather the here presented functional scheme to construct *external* vertex functions is the minimal procedure to cure the corresponding symmetry defects.

VI. CONCLUSION AND OUTLOOK

In this paper we have analyzed Baym's Φ-derivable approximations with respect to their symmetry properties. It was shown that the self-consistent mean field and propagator solutions of such approximations in general do not fulfill the Ward-Takahashi identities of the full propagator as derived from the usual 1PI generating functional, i.e., the effective action. The reason for that lies in the fact that, although the expansion of the *2pI-functional* is done systematically in a symmetry-conserving ordering scheme (for instance in powers of some coupling constant of a symmetric term in the Lagrangian, the $\hbar$- or the $1/N$-expansion) the *solutions* of the equations of motion correspond to an incomplete resummation to any order of the expansion parameter. In general not even crossing symmetry is respected for the solutions beyond the order of the expansion parameter.

Furthermore it could be shown, though, that for any such truncated Dyson resummation scheme it is always possible to define a non-perturbative 1PI-effective action based on the self-consistent solution. This supplementary action indeed generates proper n-point vertex functions which fulfill the whole hierarchy of Ward-Takahashi identities. These *external proper vertex functions* are implicitly determined by *closed* vertex equations of Bethe-Salpeter or higher order type, where all ingredients are constructed from the self-consistent mean field and the self-consistent propagator resulting from the underlying Φ-derivable Dyson scheme. The corresponding solutions exactly recover the crossing symmetry and at the same time the symmetries of the original classical action provided there are no intervening anomalies. The fact that the Φ-functional formalism was used to determine the self-consistent mean fields and propagators does not only avoid double counting but also ensures the consistency of counter terms, needed for renormalization of the divergent integrals. This is valid for both, for the 2PI functional $\Gamma[\vec{\varphi}, G]$ and its equations of motion as well as for the here discussed higher order vertex equations like the Bethe-Salpeter equation resulting from the effective quantum action $\tilde{\Gamma}[\vec{\varphi}]$. In the simple Hartree approximation the symmetry repairing procedure leads to the well known Random Phase Approximation (RPA). Similar features result for the leading order $1/N$ expansion of the 2PI functional. For approximation levels with genuine two-point self energies, i.e., for problems with damping, one arrives at Bethe-Salpeter equations of ladder type, as they were considered, e.g., in the context of Hard Thermal Loop (HTL) resummations or of photon production [22], which in semi-classical approximation leads to classical transport equations.

Although the effective 1PI-action formally fulfills all the symmetry properties of the original action it now suffers other defects essentially resulting from the lack of self-consistency for the so constructed *external* multi-point functions. Their internal structure is inherited from the self-consistent (e.g., Hartree) propagators: Since the latter do not fulfill the Ward-Takahashi identities it is not guaranteed that they show the correct dynamical behavior. For instance in the case of the here considered O(N)-model in the spontaneously broken Nambu-Goldstone phase without explicit symmetry breaking the transverse ("pionic") degrees of freedom have a non-zero mass in the self-consistent propagator and thus all "pion" loops show a wrong dynamical behavior (e.g., the vacuum "σ" spectral function shows a threshold at $s = 2M_\perp^2$ which is finite rather than zero!). Another consequence of this symmetry problem at the internal level is the wrong prediction of a first order phase transition.

Both, the here considered 1PI and the 2PI effective actions $\tilde{\Gamma}[\varphi]$ and $\Gamma[\varphi, G]$ have the same values at the solution of the self-consistent solutions for the mean fields and propagators and therefore represent a non-perturbative approximation for the thermodynamical potential. This implies that the symmetry violation at the internal level may also leaves its traces in this thermodynamical potential.

The derivation of approximation schemes that fulfill all symmetry properties of the underlying classical action and at the same time are fully self-consistent still remains as an open task.

In a forthcoming publication we shall discuss the symmetry properties for the specially interesting case of local gauge symmetries. Here the background field method permits to establish an explicitly gauge-invariant non-perturbative effective action on the basis of the 2PI-action [16], which leads to vertex functions that formally fulfill the Ward-Takahashi identities. In this case, however, the symmetry violation by the intermediate 2PI approximations causes even more serious complications, namely the excitation of unphysical degrees of freedom for the gauge field propagators. The latter fact may imply artifacts in the external proper vertex functions which among others even may spoil thermodynamic consistency due to the false number of degrees of freedom in the internal lines. Here appropriate projection methods integrated into the Dyson scheme as recently suggested by us [23] may establish a suitable work-around for such problems. For a more general recent review on Schwinger Dyson equation approaches to non-abelian gauge theories, in particular applied to the low-energy properties of QCD, see [24].

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APPENDIX A: CALCULATION OF THE 2PI FUNCTIONAL

In order to give a precise meaning for the approximation schemes considered we briefly derive the Feynman rules for the 2PI functional in terms of path integrals. The main line of arguments follows [3]. In order to perform a systematic loop expansion here we explicitely shall keep track of $\hbar$ factors.

We shall restrict ourselves to theories with only scalar boson fields. The generalization to other field contents is straight forward using the very same functional integral techniques. Thus we assume the theory to be defined in terms of a classical action functional

$$S[\phi] = \int_{\mathcal{C}} \mathrm{d}(1)\, \mathscr{L}(\phi_1), \tag{A1}$$

where ϕ denotes a multiplet of scalar fields. Here and in the following we use the notation introduced in I: $\int_{\mathcal{C}} \mathrm{d}(1 \ldots n)\ f(1, 2, \ldots, n)$ denotes the d-dimensional integral in the sense of dimensional regularization with the 0-component running along the extended Schwinger-Keldysh path $\mathcal{C}$ (running from the initial time t_i to a final time t_f and back along the real time axis and then down to $t_i - \mathrm{i}\beta$ parallel to the imaginary time axis and at the end taken $t_i \to -\infty$ and $t_f \to +\infty$). Here this contour integral also implies the appropriate sums over charge-space indices.

The generating functional with local and bilocal sources is defined by

$$Z[J, B] = N \int \mathrm{D}\phi \exp\left[\frac{\mathrm{i}}{\hbar} S[\phi] + \frac{\mathrm{i}}{\hbar} \int_{\mathcal{C}} \mathrm{d}(1)\ J_1 \phi_1 + \frac{\mathrm{i}}{2\hbar} \int_{\mathcal{C}} \mathrm{d}(12)\ B_{12} \phi_1 \phi_2 \right], \tag{A2}$$

where N is a indefinite normalization constant which will be chosen such that for the temperature $T \to 0$ and $J = B = 0$ the functional is normalized to 1.

Shifting the integration variable of the path integral by an arbitrary field φ one obtains

$$\begin{aligned} Z[J, B] =& N \exp\left[\frac{\mathrm{i}}{\hbar} S[\varphi] + \frac{\mathrm{i}}{\hbar} \int_{\mathcal{C}} \mathrm{d}(1)\ J_1 \varphi_1 + \frac{\mathrm{i}}{2\hbar} \int_{\mathcal{C}} \mathrm{d}(12)\ B_{12} \varphi_1 \varphi_2 \right] \times \\ & \times \underbrace{\int \mathrm{D}\phi' \exp\left[\frac{\mathrm{i}}{2\hbar} \int_{\mathcal{C}} \mathrm{d}(12)\ (G^{-1})_{12} \phi'_1 \phi'_2 + \frac{\mathrm{i}}{\hbar} S_I[\phi', \varphi] + \frac{\mathrm{i}}{\hbar} \int_{\mathcal{C}} \mathrm{d}(1)\ J'_1 \phi'_1 \right]}_{Z_1[J']}, \end{aligned} \tag{A3}$$

where we introduced the following abbreviations

$$\begin{aligned} (G^{-1})_{12} &= \frac{\delta^2 S[\varphi]}{\delta\varphi_1 \delta\varphi_2} + B_{12} := (\mathscr{D}^{-1})_{12} + B_{12}, \\ S_I[\phi', \varphi] &= S[\phi' + \varphi] - S[\varphi] - \int_{\mathcal{C}} \mathrm{d}(1)\ \frac{\delta S[\varphi]}{\delta\varphi_1} \phi'_1 - \frac{1}{2} \int_{\mathcal{C}} \mathrm{d}(12)\ \frac{\delta^2 S[\varphi]}{\delta\varphi_1 \delta\varphi_2} \phi'_1 \phi'_2, \\ J'_1 &= J_1 + \frac{\delta S[\varphi]}{\delta\varphi_1} + \int_{\mathcal{C}} \mathrm{d}(1')\ B_{11'} \varphi_{1'}. \end{aligned} \tag{A4}$$

Now we want to chose J and B such that φ and G are the exact mean field and the exact propagator respectively. These are defined with help of $W[J, B] = -\mathrm{i}\hbar \ln Z[J, B]$ via

$$\varphi_1 = \frac{\delta W[J, B]}{\delta J_1}, \ \mathrm{i}\hbar G_{12} = 2 \frac{\delta W[J, B]}{\delta B_{12}} - \varphi_1 \varphi_2. \tag{A5}$$

From the first condition we derive immediately that we have to chose $J' = J'_0$ such that

$$\left. \frac{\delta Z_1[J']}{\delta J'} \right|_{J' = J'_0} \stackrel{!}{=} 0. \tag{A6}$$

In this way the functional $Z_1[J'_0]$ defines completely $Z[J, B]$ and can be calculated approximately with well-known standard techniques (see, e.g., [18]). An example will be given in Sect. IV. In this way any expansion with respect to the number of loops (powers of $\hbar$) or with respect to the coupling constant defines a Φ-derivable approximation. Of course any other expansion scheme, known from usual perturbation theory, is feasible. E.g., recently in [25] and [14] investigations of the Large-N-expansion description for Baym's functional were undertaken.

For sake of completeness we give the derivation of Baym's Φ-functional. To that end we only have to perform the $\hbar$-expansion up to first order, i.e., up to one-loop order in terms of diagrams. For this task it is convenient to introduce the new action

$$\tilde{S}[\phi'; J'] = \frac{1}{2} \int_{\mathcal{C}} \mathrm{d}(12)\ (G^{-1})_{12} \phi'_1 \phi'_2 + S_I[\phi', \varphi] + \int_{\mathcal{C}} \mathrm{d}(1)\ J'_1 \phi'_1 \tag{A7}$$

so that the functional Z_1, defined in (A3), reads

$$Z_1[J'] = N \int \mathrm{D}\phi \exp\left[\frac{\mathrm{i}}{\hbar}\tilde{S}[\phi'; J']\right] = \exp\left(\frac{\mathrm{i}}{\hbar} W_1[J']\right). \tag{A8}$$

To obtain the $\hbar$-expansion we have to expand the functional integral around the solution of the classical field equations φ_0', which is the stationary point of the classical action:

$$\frac{\delta \tilde{S}[\varphi_0'; J']}{\delta \varphi_{01}'} = 0 \tag{A9}$$

and substitute $\sqrt{\hbar}\phi'' = \phi' - \varphi_0'$:

$$Z_1[J'] = N \exp\left\{\frac{\mathrm{i}}{\hbar}\tilde{S}[\varphi_0', J']\right\} \int \mathrm{D}\phi'' \exp\left[\frac{\mathrm{i}}{2}\int_{\mathcal{C}} \mathrm{d}(12)\; (\mathscr{G}^{-1})_{12}\phi_1''\phi_2'' + \frac{\mathrm{i}}{\hbar}\tilde{S}_I[\sqrt{\hbar}\phi'', \varphi_0']\right] \tag{A10}$$

with the definitions

$$\begin{aligned}
(\mathscr{G}^{-1})_{1j,2k} &= \frac{\delta^2 \tilde{S}[\varphi_0'; J']}{\delta\varphi_{01}\delta\varphi_{02}}, \\
\tilde{S}_I[\sqrt{\hbar}\phi'', \varphi_0'] &= \tilde{S}[\varphi_0' + \sqrt{\hbar}\phi''; J'] - \tilde{S}[\sqrt{\hbar}\varphi_0'; J'] - \frac{\hbar}{2}\int_{\mathcal{C}} \mathrm{d}(12)\; (\mathscr{G}^{-1})_{12}\phi_1''\phi_2''.
\end{aligned} \tag{A11}$$

Note that $(\mathscr{G}^{-1})_{12}$ and $\tilde{S}_I$ both depend on J' only implicitly via φ_0' and that $\tilde{S}_I^{(k)}[\sqrt{\hbar}\varphi'', \varphi_0] = O[\hbar^{k/2}]$ where $\tilde{S}_I^{(k)}$ denotes the monomial to order ϕ''^k. By definition only the terms with $k \geq 3$ are different from 0. The most general renormalizable theory has only $k = 3$ and $k = 4$ contributions in $\tilde{S}_I$ and we shall restrict ourselves to this case.

Now it is easy to extract the one-loop contribution (i.e., the $O(\hbar)$-contribution) to the generating functional W' for connected diagrams:

$$\begin{aligned}
W_1[J'] &= -\mathrm{i}\hbar\left[\ln N + \frac{\mathrm{i}}{\hbar}\tilde{S}[\varphi_0', J'] + \ln\int \mathrm{D}\phi'' \exp\left[\frac{\mathrm{i}}{2}\mathscr{G}_{12}^{-1}\phi_1''\phi_2''\right] + W_2[J']\right] \\
&= \tilde{S}[\varphi_0', J'] + \frac{\mathrm{i}\hbar}{2}\operatorname{Tr}\ln(\mathscr{G}_{12}^{-1}/M^2) + W_2'[J'].
\end{aligned} \tag{A12}$$

Herein we have introduced an arbitrary mass scale M to avoid dimensionful quantities within the logarithm which takes account of the overall normalization of Z which is irrelevant for any physical quantity derived from it. Now according to (A6) we have to chose $J' = J_0'$ such that $\varphi' = \delta W_1/\delta J' = 0$, and we have

$$\varphi_1' := \frac{\delta \tilde{S}[\varphi_0', J']}{\delta J_1'} + O(\hbar), \tag{A13}$$

so that we can substitute φ' instead of φ_0' in (A12) leading only to a modification of the functional W_2' to order $O(\hbar^2)$ while the $O(\hbar)$-part remains unchanged.

According to (A6) we have to chose $J' = J_0'$ such that $\varphi' = 0$ to obtain the original W-functional. We also note that for this choice $\mathscr{G}$ becomes G according to (A3):

$$W[J,B] = -\mathrm{i}\hbar\ln Z[J,B] = S[\varphi] + \int_{\mathcal{C}} \mathrm{d}(1)\; J_1\varphi_1 + \frac{1}{2}\int_{\mathcal{C}} \mathrm{d}(12)\; B_{12}\varphi_1\varphi_2 + W_1[J_0']. \tag{A14}$$

Now we define the *2PI effective action* by the double Legendre transformation of W with respect to J and B. Using (A5) this leads to

$$\begin{aligned}\Gamma[\varphi,G] &= W[J,B] - \int_{\mathcal{C}} \mathrm{d}(1)\ J_1\varphi_1 - \frac{1}{2}\int_{\mathcal{C}} \mathrm{d}(12)\ (\varphi_1\varphi_2 + \mathrm{i}\hbar G_{12})B_{12} \\ &= S[\varphi] + \frac{\mathrm{i}\hbar}{2}\operatorname{Tr}\ln(G^{-1}/M^2) + \frac{\mathrm{i}\hbar}{2}\int_{\mathcal{C}} \mathrm{d}(12)\ (\mathscr{D}^{-1})_{12}(G_{12} - \mathscr{D}_{12}) + \Phi[\varphi,G].\end{aligned} \tag{A15}$$

From the derivation we note that $\Phi[\varphi,G] = O(\hbar^2)$, i.e., in the language of diagrams it contains only diagrams with at least two loops. The main difference to the usual *1PI effective action* is that the lines appearing in the diagrams symbolize full propagators $\mathrm{i}G$ rather than perturbative ones.

The equations of motion are now given by the fact that we like to study the theory for vanishing auxiliary sources J and B. From the Legendre transformation (A14) we can immediately express this in terms of the functional Γ:

$$\begin{aligned}\frac{\delta\Gamma[\varphi,G]}{\delta\varphi} &= -J_1 - \int_{\mathcal{C}} \mathrm{d}(2)\ B_{12}\varphi_{12} \stackrel{!}{=} 0 \\ \frac{\delta\Gamma[\varphi,G]}{\delta G} &= -\frac{\mathrm{i}\hbar}{2}B_{12} \stackrel{!}{=} 0.\end{aligned} \tag{A16}$$

Using the $\hbar$-expansion (A15) the last line reads:

$$\mathscr{D}_{12}^{-1} - G_{12}^{-1} = \frac{2\mathrm{i}}{\hbar}\frac{\delta\Phi[\varphi,G]}{\delta G_{12}} := \Sigma_{12}. \tag{A17}$$

It is clear that Σ_{12} is the *exact self-energy* expressed in terms of the *exact connected propagator* G and the *exact mean field* φ and thus (A17) is the full self-consistent *Dyson equation.* This implies that no propagator line in the diagrams must contain any self-energy insertion, because these lines denote already the full propagator. Thus $\Phi[\varphi,G]$ consists of all closed diagrams with point vertices from the action $S_I[\varphi,\phi]$ with at least two loops[8], which have the additional property that it is impossible to disjoin them by cutting only one or two lines, i.e., all diagrams contained on Φ must be *two-particle irreducible* (2PI). This must hold true because taking the derivative of Φ with respect to G according to (A17) defines the full proper self-energy which must be 1PI and no propagator line should contain any self-energy insertion. Now taking this derivatives diagrammatically means to open any propagator line of all diagrams contained in Φ and taking the sum of the so obtained skeleton self-energy diagrams. This indeed implies the 2PI property for the diagrams defining Φ since otherwise one could disjoin a diagram by cutting two lines, leading to a 1PR diagram contributing to the self-energy which by definition cannot be contained in the diagram expansion of the self-energy.

In Sect. IV on the analytical example for the O(N)-linear σ-model up to order $\hbar^2$ we see that the double Legendre transformation (A15) indeed leads to 2PI diagrams for $\Phi[\varphi,G]$. In this section also an example of diagrammatical derivations for the Φ-functional for the same theory is given.

APPENDIX B: SYMMETRIES OF THE CLASSICAL ACTION

For sake of completeness we summarize the derivation of Noether's theorem [26] for symmetries of a classical action functional. For sake of simplicity we shall restrict ourselves again to a multiplet of scalar fields. The extension to more general cases is straight forward.

[8] Note that ϕ denotes the "quantum field", integrated over within the path integral and φ the "classical background field". S_I consists of the sum over all monomials with at least three quantum fields.

We investigate the behavior of the classical action functional $S[\vec{\phi}]$ under a general infinitesimal transformation of the form

$$x'^{\mu} = x^{\mu} + \delta x^{\mu}, \ \vec{\phi}'(x') = \vec{\phi}(x) + \delta\vec{\phi}(x), \tag{B1}$$

where $\delta\vec{\phi}$ may depend on both, the fields and the space-time argument. The action functional is defined to be symmetric under the transformation (B1) if its variation

$$\delta S[\vec{\phi}] = \int \mathrm{d}^d x' \mathscr{L}(\vec{\phi}', \partial'_{\mu}\vec{\phi}', x') - \int \mathrm{d}^d x \mathscr{L}(\vec{\phi}, \partial_{\mu}\vec{\phi}, x) \equiv 0. \tag{B2}$$

vanishes *identically*, i.e., without any restrictions on the fields. To derive explicit conditions for $\mathscr{L}$ to fulfill the symmetry condition (B2) we have to rewrite the first integral in terms of x, where we have to take into account the Jacobian of the volume element. In linear order of the variation we have

$$\det\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right) = 1 + \partial_{\mu}\delta x^{\mu}. \tag{B3}$$

The transformation and differentiation with respect to the space-time arguments does not commute:

$$\delta(\partial_{\mu}\phi) := \partial'_{\mu}\phi'(x') - \partial_{\mu}\phi = \partial_{\mu}(\delta\phi) - (\partial_{\mu}\delta x^{\nu})\partial_{\nu}\phi. \tag{B4}$$

After some algebraic manipulations we find

$$\delta S[\phi] = \int \mathrm{d}^n x \frac{\delta S[\phi]}{\delta\phi}\left[\delta\phi - (\partial_{\nu}\phi)\delta x^{\nu}\right] \equiv 0. \tag{B5}$$

Let now $\delta\eta^a$ be the independent parameters of the Lie group acting on the fields and the space-time variables, i.e., (B1) reads

$$\delta\phi(x) = \tau_a(x,\phi)\delta\eta^a, \ \delta x^{\mu} = -\tilde{\tau}_a^{\mu}(x,\phi)\delta\eta^a. \tag{B6}$$

This means that the symmetry condition (B5) reads

$$\int \mathrm{d}^n x \frac{\delta S[\phi]}{\delta\phi(x)}\left\{\tau_a(x,\phi) + [\partial_{\nu}\phi(x)]\tilde{\tau}_a^{\nu}(x,\phi)\right\}\delta\eta^a \equiv 0. \tag{B7}$$

Since this must hold for *any field configuration* for which the action is well defined and the $\delta\eta^a$ are independent generators of the group operation for each a there must exist a current j^a, the *Noether currents* corresponding to the symmetry group, such that

$$\frac{\delta S[\phi]}{\delta\phi(x)}\left\{\tau_a(x,\phi) + [\partial_{\nu}\phi(x)]\tilde{\tau}_a^{\nu}(x,\phi)\right\} = \partial_{\mu}j_a^{\mu}. \tag{B8}$$

Now the classical field equations of motion are given by the stationarity of the action. Thus *for the solutions of the equations of motion the Noether-currents* are conserved.

To find the explicit expression for the Noether-currents we go back to (B5) and express it in terms of the Lagrangian. After some calculations we obtain

$$\delta\eta^a\partial_{\mu}j_a^{\mu} = \partial_{\mu}\left[\left((\partial_{\nu}\phi)\frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\phi)} - \delta_{\nu}^{\mu}\mathscr{L}\right)\delta x^{\nu} - \frac{\partial\mathscr{L}}{\partial(\partial_{\mu}\phi)}\delta\phi\right] + \delta\mathscr{L} + \mathscr{L}\partial_{\mu}\delta x^{\mu}. \tag{B9}$$

This means that there must exist local functionals $\Omega_a^\mu(\phi, x)$ such that

$$\delta\mathscr{L} + \mathscr{L}\partial_\mu \delta x^\mu = \partial_\mu \Omega_a^\mu \delta\eta^a, \tag{B10}$$

which leads to the desired explicit expression for the conserved Noether current:

$$\delta\eta^a j_a^\mu = \left((\partial_\nu \phi) \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} - \delta_\nu^\mu \mathscr{L} \right) \delta x^\nu - \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \delta\phi + \Omega_a^\mu \delta\eta^a. \tag{B11}$$

It should be noted that within special relativity only the total conserved quantities are physical observables, not the local currents themselves. Those follow from the continuity equation $\partial_\mu j^\mu = 0$ by integration over a space-like closed hypersurface of space-time. Taking the special hypersurface given in a Lorentz reference frame by $x^0 = t_0$ and $x^0 = t_1$ we find

$$Q(t_1) - Q(t_0) = \int_{t_0}^{t_1} \mathrm{d}x^0 \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\, \partial_0 j^0(\vec{x}) = \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} \,\mathrm{div}\, \vec{j} = 0, \tag{B12}$$

where we have made use of the continuity equation. Eq. (B12) tells us that the *total Noether charge*

$$Q(t) = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} j^0 = \text{const.} \tag{B13}$$

For sake of completeness it should be mentioned that the currents (B11) are not determined uniquely by the symmetries since changing it according to

$$(j')^\mu = \partial_\rho \omega^{\rho\mu} \text{ with } \omega^{\rho\mu} = -\omega^{\mu\rho} \tag{B14}$$

does not change the Noether charge (B13), and for j' the continuity equation holds true as well as for j. This notion is important especially in the context of particles of higher spin where the freedom of choice of the Noether current, in (B14) parameterized by the antisymmetric tensor $\omega^{\rho\mu}$, can be used to give gauge invariant definitions of the energy momentum tensor. In general its canonical version

$$\Theta^{\mu\nu} = \partial^\mu \frac{\partial \mathscr{L}}{\partial(\partial_\nu \phi)} - g^{\mu\nu} \mathscr{L} \tag{B15}$$

is not gauge invariant. It is clear from (B11) that (B15) is the "Noether current" of the symmetry of the physical laws against space-time translations which leads to energy-momentum conservation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \mathrm{d}^3\vec{x}\, \Theta^{0\nu}(x) = 0. \tag{B16}$$

APPENDIX C: SOME FEYNMAN INTEGRALS

In this appendix we give three dimensionally regularized Feynman integrals, needed for the application to the tadpole approximation and the corresponding RPA-summed external self-energies. The techniques to obtain them can be found in standard textbooks, for instance in [27]. As usual we set $d = 4 - 2\epsilon$ for the space-time dimension in the sense of dimensional regularization, μ for the regularization scale; $\gamma \approx 0.577$ stands for the Euler-Mascheroni constant.

The first integral is the vacuum tadpole with a free propagator for a mass m:

$$\Theta_1(m^2) = \mathrm{i} \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{\mu^{2\epsilon}}{l^2 - m^2 + \mathrm{i}\eta} = \frac{m^2}{16\pi^2} \left[-\frac{1}{\epsilon} - 1 + \gamma + \ln\left(\frac{m^2}{4\pi\mu^2} \right) \right] + O(\epsilon). \tag{C1}$$

The next integral is used to renormalize the divergent four-point sub-diagrams contained in the tadpole approximation as a hidden divergence:

$$\begin{aligned}\Theta_2(m^2) &= \mathrm{i}\int \frac{\mathrm{d}^d l}{(2\pi)^d}\frac{\mu^{2\epsilon}}{(l^2-m^2+\mathrm{i}\eta)^2} = \frac{1}{2m}\partial_m\Theta_1(m^2)\\ &= \frac{1}{16\pi^2}\left[-\frac{1}{\epsilon}+\gamma+\ln\left(\frac{m^2}{4\pi\mu^2}\right)\right]+O(\epsilon).\end{aligned} \tag{C2}$$

Further for the calculation of the external self-energy we need the two-point function

$$\begin{aligned}L_{m_1,m_2}(p^2) &= \mathrm{i}\int\frac{\mathrm{d}^d l}{(2\pi)^d}\frac{\mu^{2\epsilon}}{(l^2-m_1^2+\mathrm{i}\eta)[(l-p)^2-m_2^2+\mathrm{i}\eta]}\\ &= \frac{1}{16\pi^2}\Bigg\{-\frac{1}{\epsilon}-2+\gamma+\frac{\lambda(p^2,m_1^2,m_2^2)}{p^2}\\ &\quad\times\left[\operatorname{artanh}\left(\frac{m_1^2-m_2^2+p^2}{\lambda(p^2,m_1^2,m_2^2)}\right)+\operatorname{artanh}\left(\frac{m_2^2-m_1^2+p^2}{\lambda(p^2,m_1^2,m_2^2)}+\right)\right]\\ &\quad+\frac{m_1^2-m_2^2}{p^2}\ln\left(\frac{m_1}{m_2}\right)+\ln\left(\frac{m_1 m_2}{4\pi\mu^2}\right)\Bigg\}+O(\epsilon),\end{aligned} \tag{C3}$$

where the Källèn function reads

$$\lambda(p^2,m_1^2,m_2^2) = \sqrt{[p^2-(m_1+m_2)^2][p^2-(m_1-m_2)^2]}. \tag{C4}$$

For the proof of Goldstone's theorem for the external propagator we need this function at $p=0$ which can be expressed with help of the already defined function Θ_1:

$$L_{m_1,m_2}(0) = \frac{1}{m_1^2-m_2^2}\left[\Theta_1(m_1^2)-\Theta_1(m_2^2)\right] \tag{C5}$$

Their expressions for equal masses read

$$\begin{aligned}L_{m,m}(p) = \frac{1}{16\pi^2}\Bigg[&-\frac{1}{\epsilon}-2+\gamma+\ln\left(\frac{m^2}{4\pi\mu^2}\right)\\ &+2\frac{\lambda(p^2,m^2,m^2)}{p^2}\operatorname{artanh}\left(\frac{p^2}{\lambda(p^2,m^2,m^2)}\right)\Bigg]\end{aligned} \tag{C6}$$

and for $p=0$ this expression becomes

$$L_{m,m}(0) = \frac{1}{2m}\partial_m\Theta_1(m^2) = \Theta_2(m^2). \tag{C7}$$

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