Off-equilibrium Bose-Einstein Condensation and 2↔3 scattering processes

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July 12, 2018

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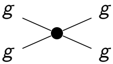
Outline

kinetic equation(1)

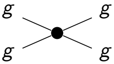
Relativistic evolution equation of a bosonic (including quantum statistics by Bose enhancement) system in non-equilibrium

$$\frac{1}{E_1}\left(p_1^{\mu}\frac{\partial}{\partial x^{\mu}} + m\frac{\partial}{\partial p_1^{\mu}}K_1^{\mu}\right) f_1 = \frac{1}{2E_1}\int \frac{d^3\vec{p_2}}{2(2\pi)^3 E_2}\frac{d^3\vec{p_3}}{2(2\pi)^3 E_3}\frac{d^3\vec{p_4}}{2(2\pi)^3 E_4} W_{12\leftrightarrow 34} \times \left\{ f_3 f_4 (1+f_1)(1+f_2) - f_1 f_2 (1+f_3)(1+f_4) \right\}$$

$=:$ 

$$W_{12\leftrightarrow 34} := (2\pi)^4 |M_{12\leftrightarrow 34}|^2 \delta^{(4)}(P_1 + P_2 - P_3 - P_4).$$

Equilibrium f_{eq}? Detailed balance!

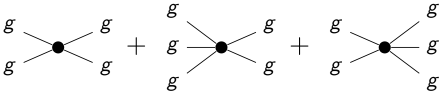


$$\overset{!}{=} 0 \rightarrow f_{\mathrm{eq}} = \frac{1}{\exp\frac{E-\mu}{T} - 1}$$

kinetic equation(2)

Relativistic evolution equation of a bosonic (including quantum statistics by Bose enhancement) system in non-equilibrium

$$\frac{1}{E_1}\left(p_1^\mu \frac{\partial}{\partial x^\mu} + m\frac{\partial}{\partial p_1^\mu} K_1^\mu\right) f_1 =$$



$$W_{123\leftrightarrow 45} := (2\pi)^4 |M_{123\leftrightarrow 45}|^2 \delta^{(4)}(P_1 + P_2 + P_3 - P_4 - P_5)$$
$$W_{12\leftrightarrow 345} := (2\pi)^4 |M_{12\leftrightarrow 345}|^2 \delta^{(4)}(P_1 + P_2 - P_3 - P_4 - P_5)$$

Equilibrium f_{eq}? Detailed balance!

$$\mu = 0\text{GeV} \qquad f_{eq} = \frac{1}{\exp\frac{E}{T} - 1}$$

- isotropic / homogeneous system $f(t, \vec{r}, \vec{p}) \to f(t, p)$
- constant crosssections:

$$
\begin{aligned}
&\left|M_{12\leftrightarrow 34}\right|^2 = 32\pi s\sigma_{22} \\
&\left|M_{12\leftrightarrow 345}\right|^2 = 192\pi^3\sigma_{23} = d\left|M_{123\leftrightarrow 45}\right|^2 \\
&d = 2 \times 8
\end{aligned}
$$

[A. El. / Nuclear Physics A 925 (2014) 150-160]

- vanishing external forces $K^\mu = 0$
- massive particles $m > 0$

Detailed balance is satisfied by the Bose-Einstein distribution

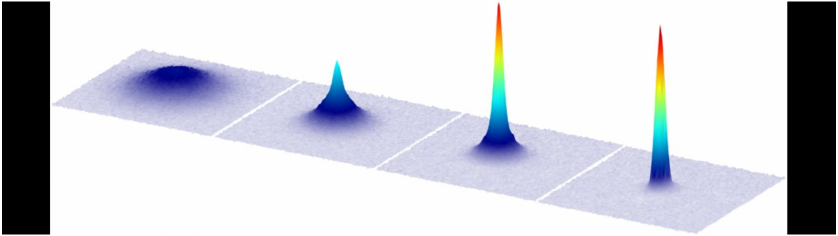
$$f_{\text{eq}}(E_i) = \frac{1}{\exp\left(\frac{E_i - \mu}{T}\right) - 1}.$$

The ground state can become macroscopically large $f_{\text{eq}}(E_0) \gg 1$.
Two cases are considered.

Decreasing the temperature

$$f(E_i) = \frac{1}{\exp\left(\frac{E_i - \mu}{T}\right) - 1} \xrightarrow{T \to 0} 0 \quad \text{for} \quad E_i > E_0 \geq \mu.$$

$\longrightarrow$ The occupation number of the ground state $f(E_0)$ becomes macroscopically large



Picture: http://www.erbium.at/FF/wp-content/uploads/2016/01/FirstErbiumBEC-1250x350.jpg

Increasing the particle density

$$f(E_0) = \frac{1}{\exp\left(\frac{E_0-\mu}{T}\right) - 1} \xrightarrow{\mu\to m} \infty \quad \text{for} \quad E_0 = m \geq \mu.$$

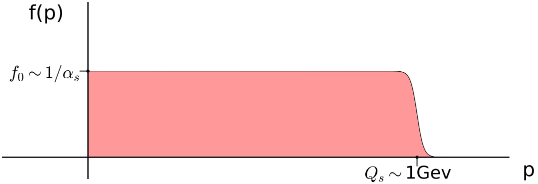
$\longrightarrow$ The occupation number of the ground state $f(E_0)$ becomes macroscopic large

Can be applied to a very early stage of heavy ion collision:

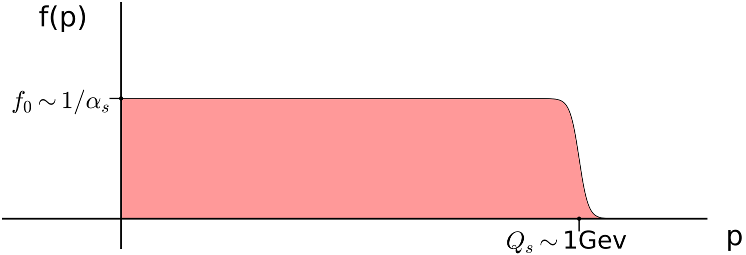
Gluons, CGC -Model

$$f(p) \sim 1/\alpha_s \text{ for } p < Q_s$$

$$\simeq f_0 \theta \left(1 - \frac{p}{Q_s}\right)$$



Determining the equilibrium state



$$n_{\text{tot}} = \int_0^\infty \frac{dp}{2\pi^2} p^2 f_{\text{init}} = \frac{f_0 Q_s^3}{6\pi^2}$$

$$\epsilon_{\text{tot}} = \int_0^\infty \frac{dp}{2\pi^2} p^2 E f_{\text{init}} = \frac{f_0}{16\pi^2} \left\{ Q_s E_{Q_s} (m^2 + 2Q_s^2) + m^4 \log \frac{m}{Q_s + E_{Q_s}} \right\}$$

$$E_{Q_s} := \sqrt{m^2 + Q_s^2}$$

Decompose $f(t,p)$ [Semikoz, Tchakev, arxiv.org/abs/hep-ph/9507306]

$$f(t,p) = f_{\text{part}}(t, p > 0) + n_c(t)(2\pi)^3\delta^{(3)}(\vec{p})$$

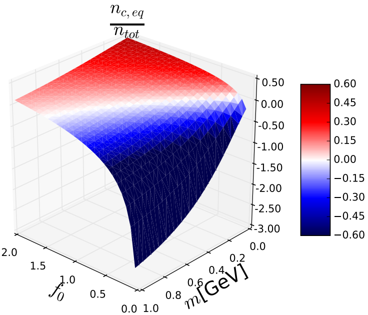
Red known — Blue unknown — Green fixing

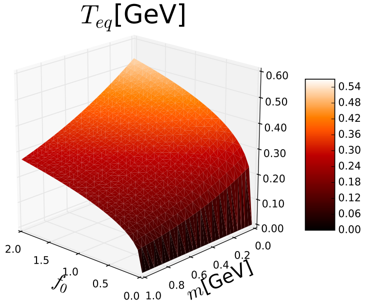
- $n_{\text{tot}} = n_{\text{part,eq}} + n_{\text{c,eq}}$ particle (density) conservation
- $\epsilon_{\text{tot}} = \epsilon_{\text{part,eq}} + \epsilon_{\text{c,eq}}$ energy (density) conservation
- $n_{\text{part,eq}} = \int_0^\infty \frac{\mathrm{d}p}{2\pi^2} p^2 \frac{1}{\exp\left(\frac{E-\mu_{\text{eq}}}{T_{\text{eq}}}\right)-1}$
- $\epsilon_{\text{part,eq}} = \int_0^\infty \frac{\mathrm{d}p}{2\pi^2} p^2 \frac{E}{\exp\left(\frac{E-\mu_{\text{eq}}}{T_{\text{eq}}}\right)-1}$
- $\mu_{\text{eq}} = m$ and $\epsilon_c = n_c m$

Solve for T_{eq} and $n_{\text{c,eq}}$

Numerical solution for the equilibrium values

- $Q_s = 1\text{GeV}$





- The blue shaded area suggest a negative condensate density which is not physical
- Condition $\mu_{\text{eq}} = m$ does not apply (underpopulated case)

massless case $m = 0$

Equilibrium state is given analytically

$$\mu_{\text{eq}} = 0, \qquad \epsilon_{\text{c,eq}} = 0$$

$$\epsilon_{\text{tot,eq}} = \frac{f_0 Q_s^4}{8\pi^2} \overset{!}{=} \frac{\pi^2 T_{\text{eq}}^4}{30} = \epsilon_{\text{part,eq}} \longrightarrow T_{\text{eq}} = \sqrt[4]{f_0 15}\frac{Q_s}{2\pi}$$

$$n_{\text{tot,eq}} = \frac{f_0 Q_s^3}{6\pi^2}, \qquad n_{\text{part,eq}} = (15 f_0)^{\frac{3}{4}} \frac{Q_s^3 \zeta(3)}{2\sqrt{2}\pi^5}$$

$$n_{\text{c,eq}} = \frac{f_0 Q_s^3}{6\pi^2} - (15 f_0)^{\frac{3}{4}} \frac{Q_s^3 \zeta(3)}{2\sqrt{2}\pi^5}$$

$$n_{\text{c,eq}} = 0 \longrightarrow f_0 \approx 0.154 \text{ (the critical case)}$$

Equilibrium for underpopulated systems

Solve for T_{eq} and μ_{eq}:

$$n_{\mathrm{tot}} = n_{\mathrm{part,eq}} = \int_0^\infty \frac{dp}{2\pi^2} p^2 f_{\mathrm{eq}}(\mu_{\mathrm{eq}}, T_{\mathrm{eq}})$$

$$\epsilon_{\mathrm{tot}} = \epsilon_{\mathrm{part,eq}} = \int_0^\infty \frac{dp}{2\pi^2} p^2 E f_{\mathrm{eq}}(\mu_{\mathrm{eq}}, T_{\mathrm{eq}})$$

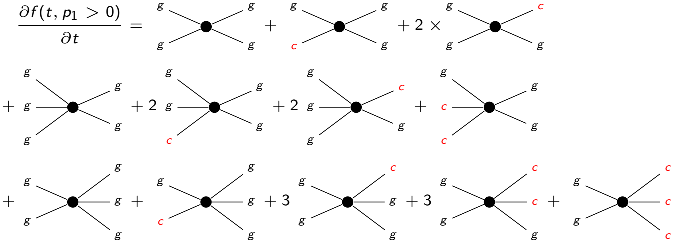
Ansatz for a marcoscopic state

$$f(t,\vec{p}) \to f(t,\vec{p}) = f_{\text{part}}(t,\vec{p}>0) + n_c(t)(2\pi)^3\delta^{(3)}(\vec{p})$$

- Set of two coupled first-order differential equations
- In this study only contributions for the massive case

Evolution equation for the distri. function (e.g. 'higher' modes)

$$\frac{\partial f(t,p_1>0)}{\partial t} =$$



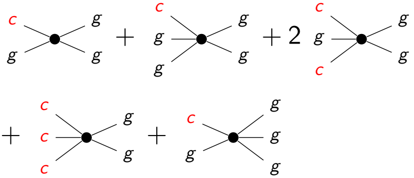
Ansatz for a marcoscopic state

$$f(t,\vec{p}) \to f(t,\vec{p}) = f_{\text{part}}(t,\vec{p}>0) + n_c(t)(2\pi)^3\delta^{(3)}(\vec{p})$$

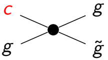
- Set of two coupled first-order differential equations
- In this study only contributions for the massive case

Evolution equation for the BEC (e.g. the 'zero' mode)

$$\int_{\mathbb{R}^3} \frac{\mathrm{d}\vec{p_1}}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\vec{p_1})\frac{\partial n_c}{\partial t} = \frac{\partial n_c}{\partial t} =$$



$|M_{12\leftrightarrow 34}|^2 \propto s = (P_1 + P_2)^2 \rightarrow$ integrate out the angular dependencies and the internal momenta p_4 with $\delta(m_1 + E_2 - E_3 - E_4)$



$$= n_c \sigma_{22} \frac{9}{64\pi^3} \int_0^\infty \mathrm{d}p_2 \int_0^\infty \mathrm{d}p_3 \frac{p_2 p_3}{m_1 E_2 E_3}$$
$$\times \left[-1 - \epsilon(p_2 - p_3 - \tilde{p}_4) + \epsilon(p_2 + p_3 - \tilde{p}_4) + \epsilon(p_2 - p_3 + \tilde{p}_4)\right]$$
$$\times \left(m_1^2 + m_2^2 + 2m_1 E_2\right) \theta(\tilde{p}_4^2)[f_3 \tilde{f}_4 - f_2(1 + f_3 + \tilde{f}_4)]$$

$$\textit{with} \quad \tilde{p}_4^2 = (m_1 + E_2 - E_3)^2 - m_4^2 > 0$$

$$\textit{and} \quad \epsilon(x) := \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

What we have so far

Given initial state with the mass of the particles m:

Final equilibrium state:

$$f_{\text{init}}(p) = f_0 \theta \left(1 - \frac{p}{Q_s}\right) \xrightarrow{EoM} f_{\text{eq}}(p) = \frac{1}{\exp\left(\frac{\sqrt{p^2+m^2}-\mu_{\text{eq}}}{T_{\text{eq}}}\right) - 1} + n_{c,\text{eq}}(2\pi)^3 \delta^{(3)}(\vec{p})$$

- Two first order coupled integro-differential equations
- Interaction described by $\sigma_{22} = \text{const}$ and $\sigma_{23} = \text{const}$
- Analytic solution? Researched field - [arxiv:1507.07834]
- Numerical evaluation!

- $f_{\text{part}}(p)$ is given on a Grid $G := \{p[0], p[1], ..., p[i], ..., p[N]\}$ with $0 = p[0] < p[1] < ... < p[i] < ... < p[N]$ and $(N > 100)$:

$$\dot{f}(p_1[0]) = C_{22}(p_1[0]) + C_{23}(p_1[0]) + C_{32}(p_1[0]) + \dots$$
$$\dot{f}(p_1[1]) = C_{22}(p_1[1]) + C_{23}(p_1[1]) + C_{32}(p_1[1]) + \dots$$
$$\vdots$$
$$\dot{f}(p_1[i]) = C_{22}(p_1[i]) + C_{23}(p_1[i]) + C_{32}(p_1[i]) + \dots$$
$$\vdots$$
$$\dot{f}(p_1[N]) = C_{22}(p_1[N]) + C_{23}(p_1[N]) + C_{32}(p_1[N]) + \dots$$

- to evaluate the collision integrals we apply quadrature methods (Simpson) for Dim=1, 2 and Monte Carlo (Vegas) for Dim=3

$$f_{i+1}^{\mathrm{RK4}} = f_i^{\mathrm{RK4}} + \frac{2825}{27648}k_{i,1} + \frac{18575}{48384}k_{i,3} + \frac{13525}{55296}k_{i,4} - \frac{277}{14336}k_{i,5} + \frac{1}{4}k_{i,6}$$
$$f_{i+1}^{\mathrm{RK5}} = f_i^{\mathrm{RK5}} + \frac{37}{378}k_{i,1} + \frac{250}{621}k_{i,3} + \frac{125}{594}k_{i,4} - \frac{1}{5}k_{i,6}$$

[Transactions on Mathematical Software 16: 201-222, 1990. doi:10.1145/79505.79507]

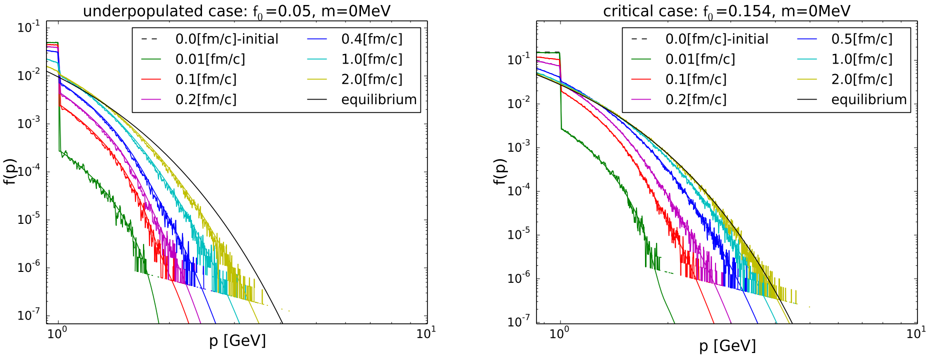
- two approximations of order 4 and 5
- no additional computation time for the second approximation
- compare the approximations
- Cash-Karp method involves $h_{\mathrm{new}} = s h_{\mathrm{old}}$

$$s = \left| \frac{\epsilon_{\mathrm{tol}}}{f_{i+1}^{\mathrm{RK5}} - f_{i+1}^{\mathrm{RK4}}} \right|^{\frac{1}{5}}$$

Condensation onset

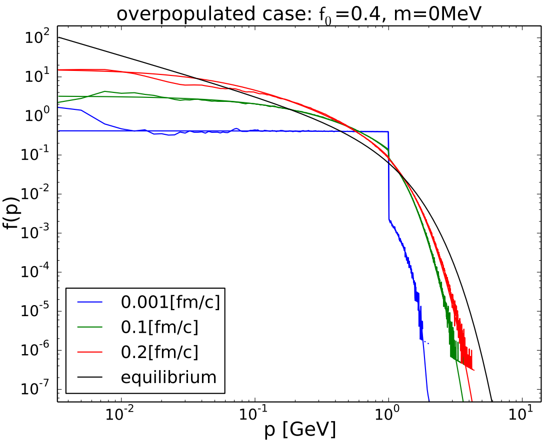
onset:= Starting time of condensation

- condensation process $\dot{n}_c \propto n_c$ only for $n_c \neq 0$
- BEC due to fluctuations $\rightarrow$ workaround
- extraction of 2 parameters $(\mu_{\text{eff}}, T_{\text{eff}})$ by fitting the Bose distribution to f_{part}.
- two possibilities to include condensation are:
  1. initialising with a finite but negligibly small condensate seed $n_c \ll n_{\text{tot}}$
  2. inserting a small condensate seed $n_c \ll n_{\text{tot}}$ when the distribution function reaches a certain point
- extraction of 2 parameters $(\mu_{\text{eff}}, T_{\text{eff}})$ by fitting the Bose distribution to f_{part} and then inserting the seed when $\mu_{\text{eff}} = m$

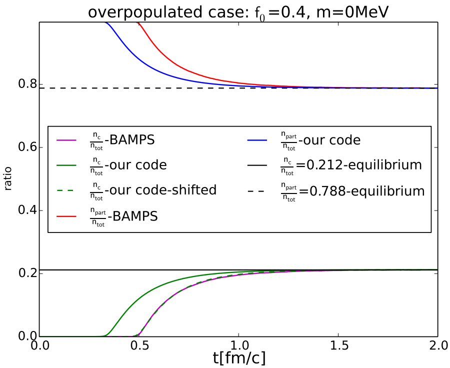


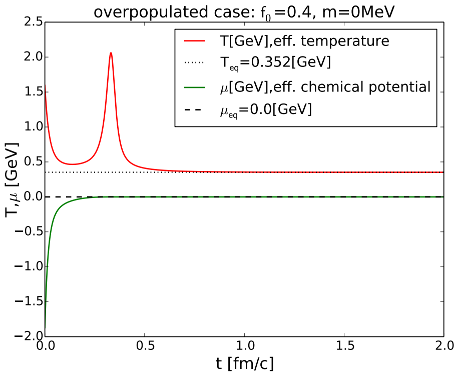
- focus $f(t = 2.0[\text{fm}/\text{c}], p)$ XXXXX
- Increasing the total particle density leads to a faster thermalisation-consistent

Overpopulated massless case

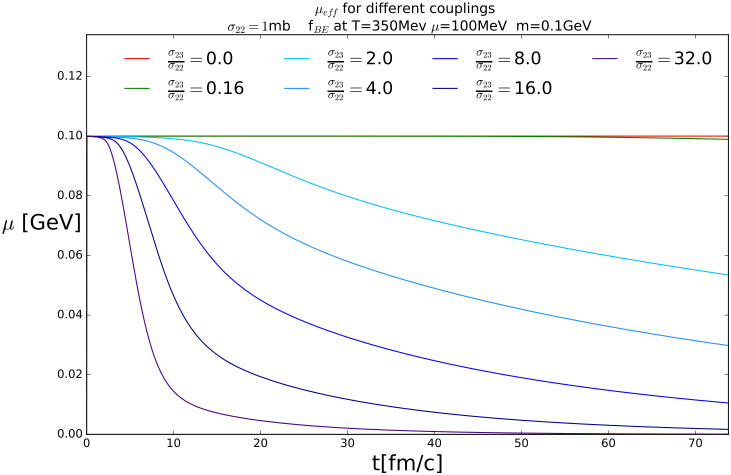


Condensate evolution

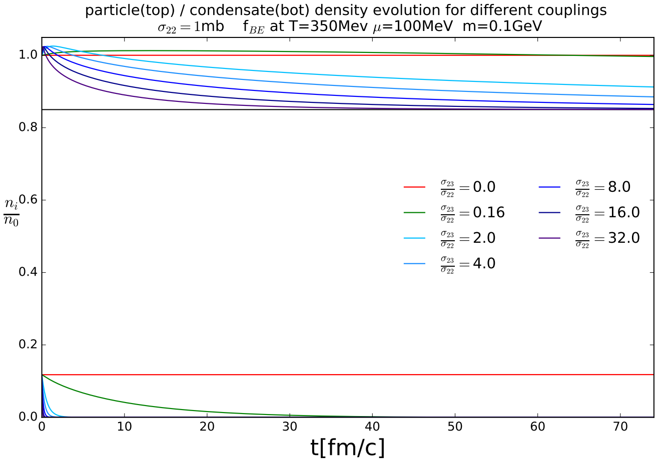




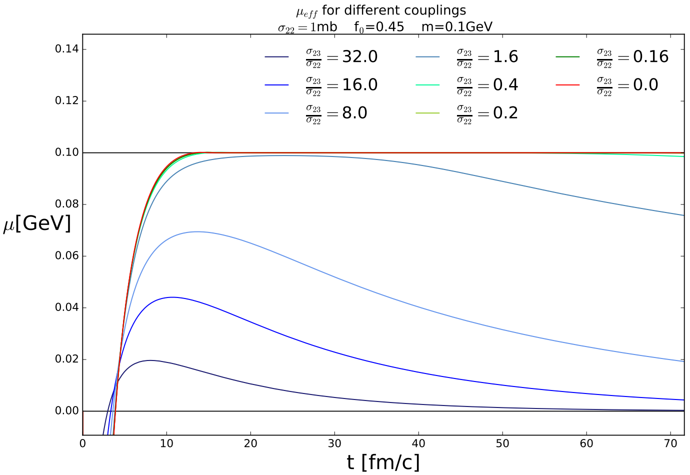
Test-scenario



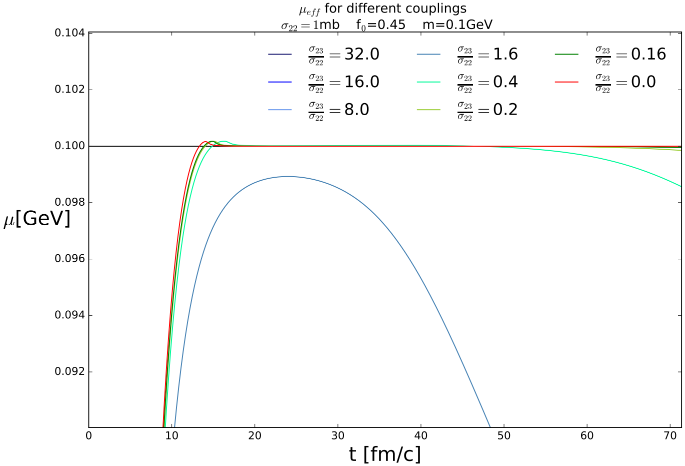
Test-scenario



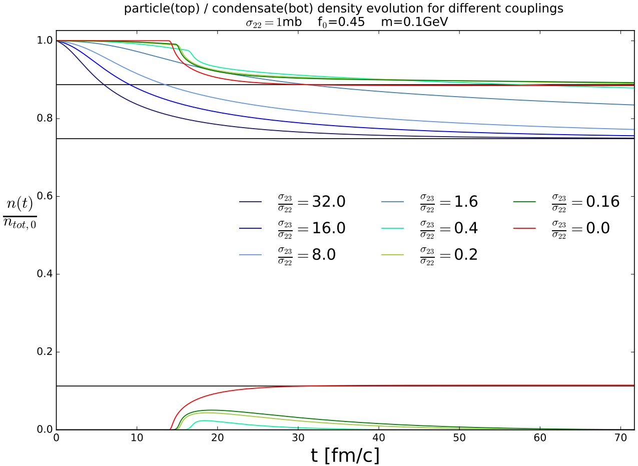
BEC formation from non equilibrium



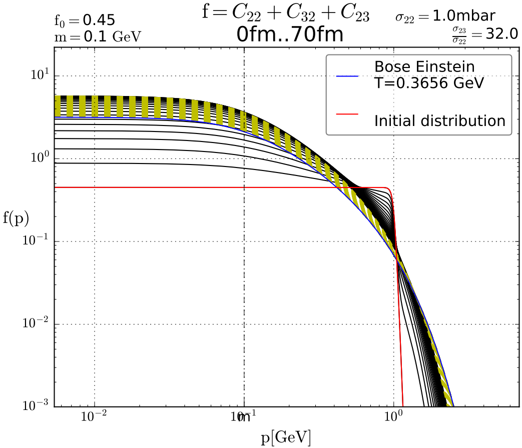
BEC formation from non equilibrium



BEC formation from non equilibrium



Overpopulated massive $m = 100 MeV$ case and high σ_{23}



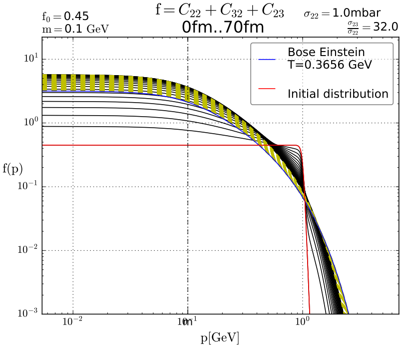
Conclusion and Outlook

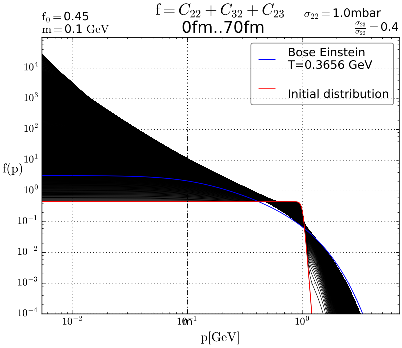
- inelastic processes limit BEC formation ultimatly for massive particles
- for low $\frac{\sigma_{23}}{\sigma_{22}}$ condensation happens, while thermalization of the system $f_{part}(p > 0)$ takes more time implying a short lived BEC- state.

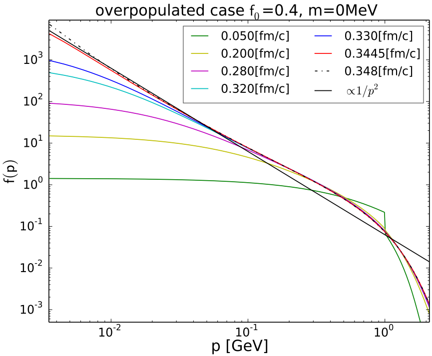
- comparison with BAMPS
- inclusion of more realistically crosssections and massless particles
- inclusion of $q, \bar{q}$
- Adapting this scheme for longitudinal expanding systems (Bjorken coordinates)

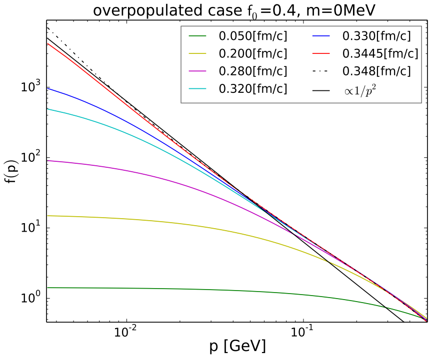
Thank You for Your attention!

Overpopulated massive $m = 100 MeV$ case with inelastice scattering






overpopulated case f_0=0.4, m=0MeV
0.050[fm/c]
0.200[fm/c]
0.280[fm/c]
0.320[fm/c]
0.330[fm/c]
0.3445[fm/c]
0.348[fm/c]
$\propto 1/p^2$
f(p)
p [GeV]


overpopulated case f_0=0.4, m=0MeV
0.050[fm/c]
0.200[fm/c]
0.280[fm/c]
0.320[fm/c]
0.330[fm/c]
0.3445[fm/c]
0.348[fm/c]
$\propto 1/p^2$
f(p)
p [GeV]

Cash-Karp RK45 -scheme

starting point: 2 approximations whereby both approximations need the evaluation of the following six values

$$k_{i,1} = h\mathcal{C}(t_i, f_i)$$

$$k_{i,2} = h\mathcal{C}\left(t_i + \frac{1}{5}h, f_i + \frac{1}{5}k_{i,1}\right)$$

$$k_{i,3} = h\mathcal{C}\left(t_i + \frac{3}{10}h, f_i + \frac{3}{40}k_{i,1} + \frac{9}{40}k_{i,2}\right)$$

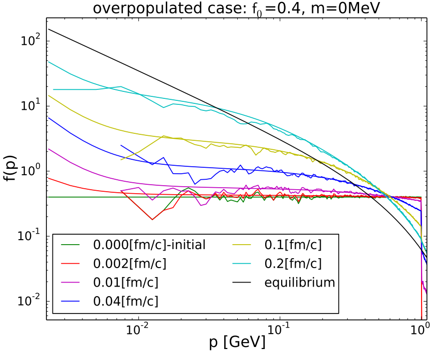
$$k_{i,4} = h\mathcal{C}\left(t_i + \frac{3}{5}h, f_i + \frac{3}{10}k_{i,1} - \frac{9}{10}k_{i,2} + \frac{6}{5}k_{i,3}\right)$$

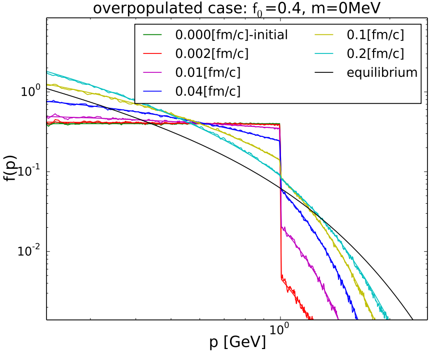
$$k_{i,5} = h\mathcal{C}\left(t_i + h, f_i - \frac{11}{54}k_{i,1} + \frac{5}{2}k_{i,2} - \frac{70}{27}k_{i,3} + \frac{35}{27}k_{i,4}\right)$$

$$k_{i,6} = h\mathcal{C}\left(t_i + \frac{7}{8}h, f_i + \frac{1631}{55296}k_{i,1} + \frac{175}{512}k_{i,2} + \frac{575}{13824}k_{i,3} + \frac{44275}{110592}k_{i,4} - \frac{253}{4096}k_{i,5}\right)$$

Overpopulated massless case $f_0 = 0.4$

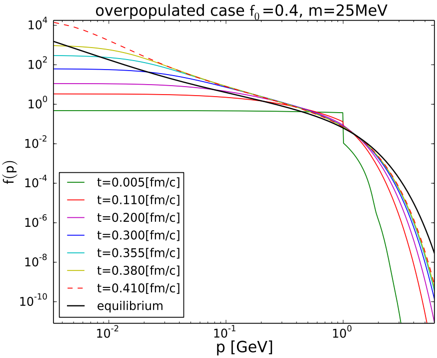
our code initialised with a seed



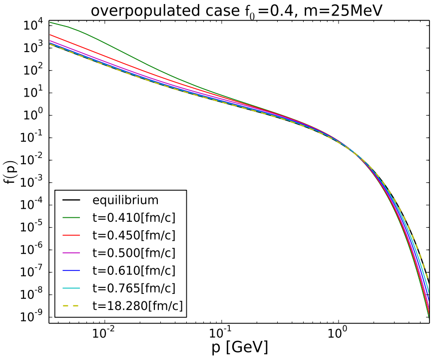


Overpopulated massive $m = 25 MeV$ case

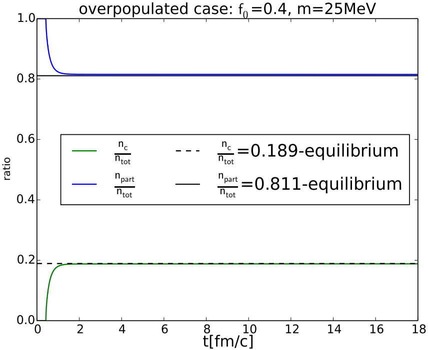
Before the onset

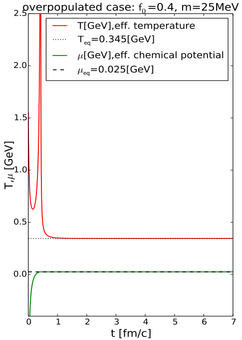


After the onset

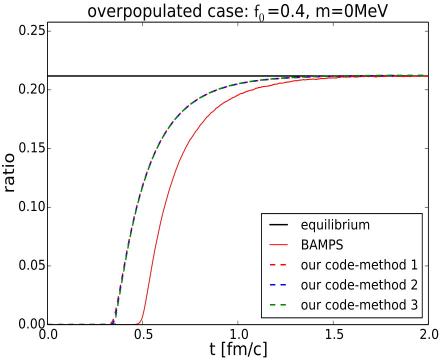


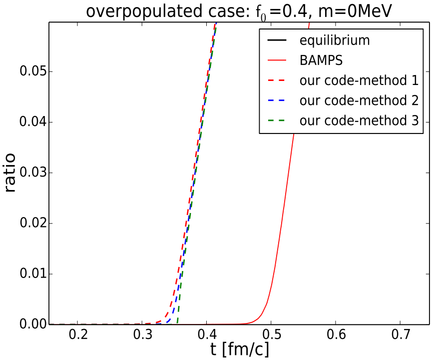
Overpopulated massive $m = 25 MeV$ case





- About the onset (onset:= Starting time of condensation)





- method 1: Starting with condensate seed
- method 2: Inserting a seed when $\mu_{\text{eff}} = 0$
- method 3: Any time