Shakhov-like extension of the relaxation-time approximation in relativistic kinetic theory

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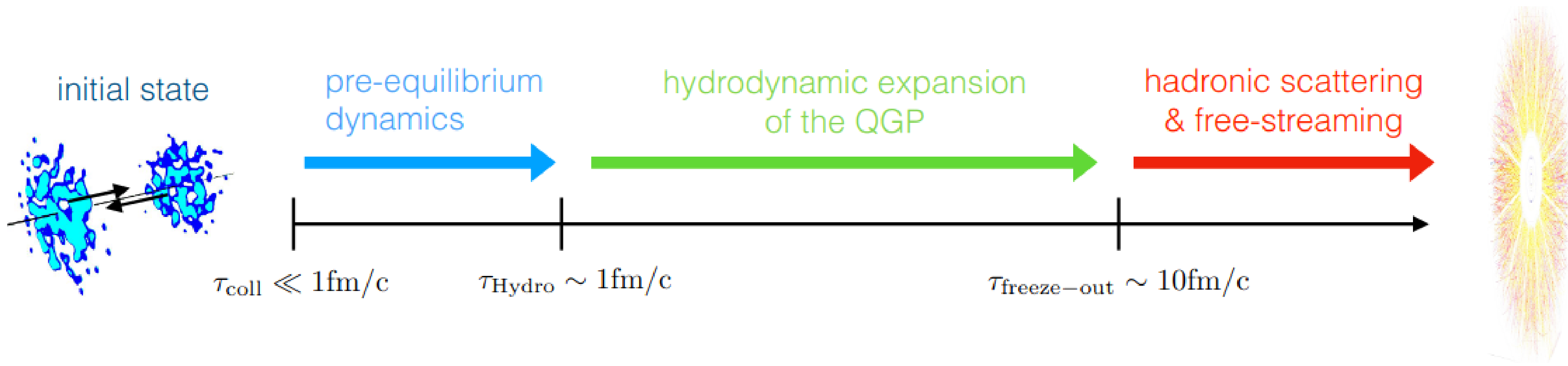
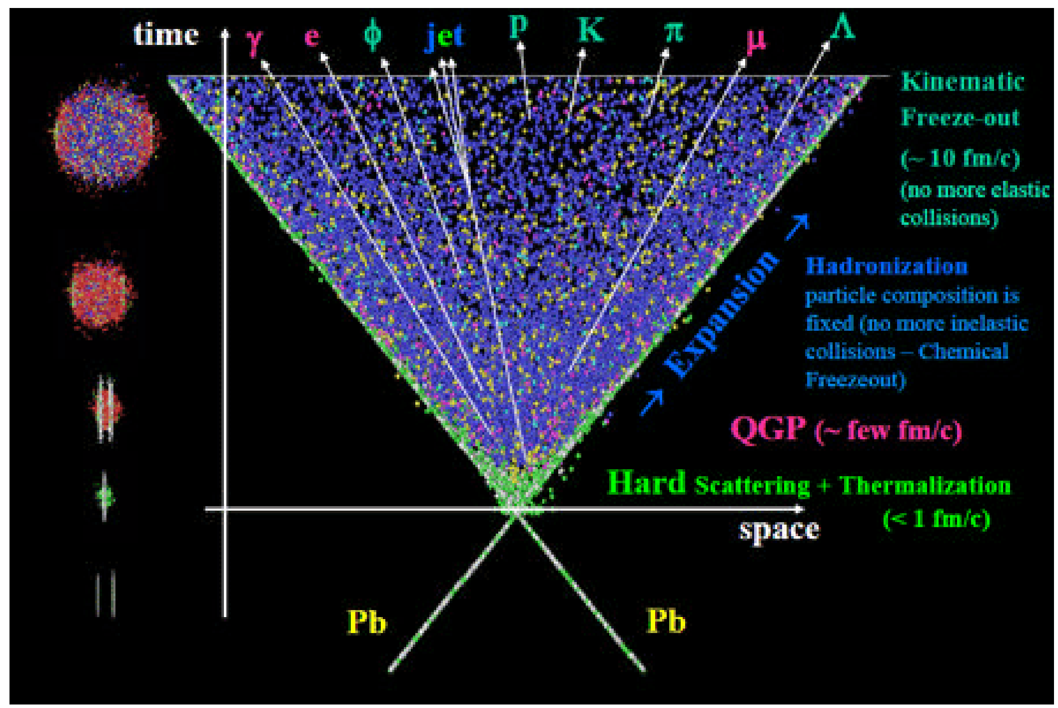
Relativistic hydro playground: Heavy-ion collisions

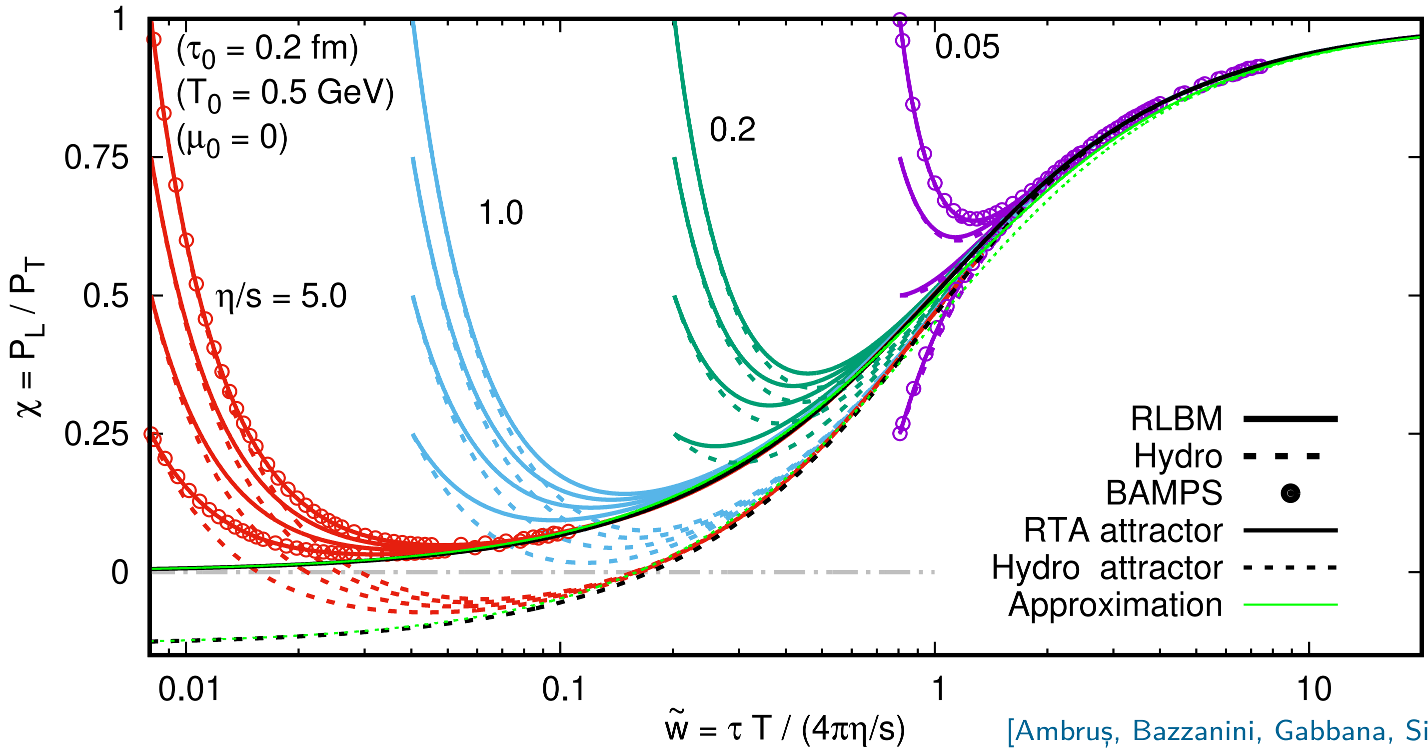


- Shortly after the collision, the system is far-from-equilibrium.
- Pre-eq. dynamics require a non-eq. description.
- Strongly-interacting QGP leaves imprints of thermalization and collectivity in final-state observables.



[Venaruzzo, PhD Thesis, 2011]

Hydro vs Kinetic theory



[Ambruș, Bazzanini, Gabbana, Simeoni, Succi, Nature Comput. Sci. **2** (2022) 641]

- Hydro employed in HIC modelling, but it breaks down far from eq.
- Kinetic theory overcomes this limitation, but realistic simulations are expensive due to $C[f]$.

  AMPT: He, Edmonds, Lin, Liu, Molnar, Wang [PLB 753 (2016) 506]
  BAMPS: Greif, Greiner, Schenke, Schlichting, Xu [PRD 96 (2017) 091504]
- RTA: $C[f] = -\frac{E_{\mathbf{k}}}{\tau_R}(f_{\mathbf{k}} - f_{0\mathbf{k}}) \Rightarrow 1-2$ o.m. faster than BAMPS.

  VEA, Busuioc, Fotakis, Gallmeister, Greiner [PRD **104** (2021) 094022]
- τ_R fixes the IR limit of RTA by matching e.g. η to that of $C[f] \Rightarrow$ good agreement with BAMPS.

Anderson-Witting model

- The Anderson & Witting RTA reads [Anderson, Witting, Physica **74** (1974) 466]

$$k^\mu \partial_\mu f_{\mathbf{k}} = C_{\mathrm{AW}}[f], \quad C_{\mathrm{AW}}[f] = -\frac{E_{\mathbf{k}}}{\tau_R}(f_{\mathbf{k}} - f_{0\mathbf{k}}), \tag{1}$$

  where $E_{\mathbf{k}} = k^\mu u_\mu$, and τ_R is the relaxation time.
- The macroscopic quantities N^μ and $T^{\mu\nu}$ are obtained from $f_{\mathbf{k}}$ via

$$N^\mu = \int dK\, k^\mu\, f_{\mathbf{k}}, \quad T^{\mu\nu} = \int dK\, k^\mu k^\nu f_{\mathbf{k}}, \tag{2}$$

  where $dK = g\, d^3k/[k_0(2\pi)^3]$ and g is the degeneracy factor.
- $f_{0\mathbf{k}}$ describes a fictitious local thermodynamic equilibrium, for which

$$N_0^\mu = n_0 u^\mu, \quad T_0^{\mu\nu} = \epsilon_0 u^\mu u^\nu - P_0 \Delta^{\mu\nu}, \tag{3}$$

  with $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$.
- Imposing $\partial_\mu N^\mu = \partial_\nu T^{\mu\nu} = 0$ requires Landau matching:

$$n = n_0, \quad \epsilon = \epsilon_0, \quad T^\mu{}_\nu u^\nu = \epsilon u^\mu. \tag{4}$$

- The AW model retains from $C[f]$ the property of driving $f_{\mathbf{k}}$ towards $f_{0\mathbf{k}}$, on a timescale τ_R.

Chapman-Enskog expansion

- We are now interested to obtain constitutive relations for the non-equilibrium quantities

$$N^{\mu} - N_0^{\mu} = V^{\mu}, \quad T^{\mu\nu} - T_0^{\mu\nu} = -\Pi\Delta^{\mu\nu} + \pi^{\mu\nu}. \tag{5}$$

- Employing the Chapman-Enskog procedure gives

$$\delta f_{\mathbf{k}} \equiv f_{\mathbf{k}} - f_{0\mathbf{k}} \simeq -\frac{\tau_R}{E_{\mathbf{k}}} k^{\mu}\partial_{\mu} f_{0\mathbf{k}} = -\tau_R f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}\big[E_{\mathbf{k}}^2\dot{\beta} - E_{\mathbf{k}}\dot{\alpha}$$
$$+\tfrac{\beta}{3}(m^2 - E_{\mathbf{k}}^2)\theta + k^{\langle\mu\rangle}(\beta E_{\mathbf{k}}\dot{u}_{\mu} + E_{\mathbf{k}}\nabla_{\mu}\beta - \nabla_{\mu}\alpha) + \beta k^{\langle\mu}k^{\nu\rangle}\sigma_{\mu\nu}\big],$$

with $\tilde{f}_{0\mathbf{k}} = 1 - af_{0\mathbf{k}}$, $\alpha = \beta\mu$, $\theta = \partial_{\mu}u^{\mu}$ and $\sigma^{\mu\nu} = \nabla^{\langle\mu}u^{\nu\rangle}$.

- Taking appropriate moments gives

$$\Pi = -\zeta_R\theta, \quad V^{\mu} = \kappa_R\nabla^{\mu}\alpha, \quad \pi^{\mu\nu} = 2\eta_R\sigma^{\mu\nu}, \tag{6}$$

where ζ_R, κ_R and η_R are given by

$$\zeta_R = \frac{m^2}{3}\tau_R\alpha_0^{(0)}, \quad \kappa_R = \tau_R\alpha_0^{(1)}, \quad \eta_R = \tau_R\alpha_0^{(2)}. \tag{7}$$

where $\alpha_0^{(\ell)}$ are τ_R-independent thermodynamic functions.

QGP Transport coefficients

- Bayesian estimation shows that η/s and ζ/s can be parametrized as

J. E. Bernhard, J. S. Moreland, S. A. Bass, Nature Phys. **15** (2019) 1113

$$\frac{\eta}{s} = (\eta/s)_{\rm min} + (\eta/s)_{\rm slope}(T - T_c)\left(\frac{T}{T_c}\right)^{(\eta/s)_{\rm crv}}, \tag{8}$$

$$\frac{\zeta}{s} = (\zeta/s)_{\rm max} \times \left[1 + \left(\frac{T - T_{\rm peak}}{(\zeta/s)_{\rm width}}\right)^2\right]^{-1}. \tag{9}$$

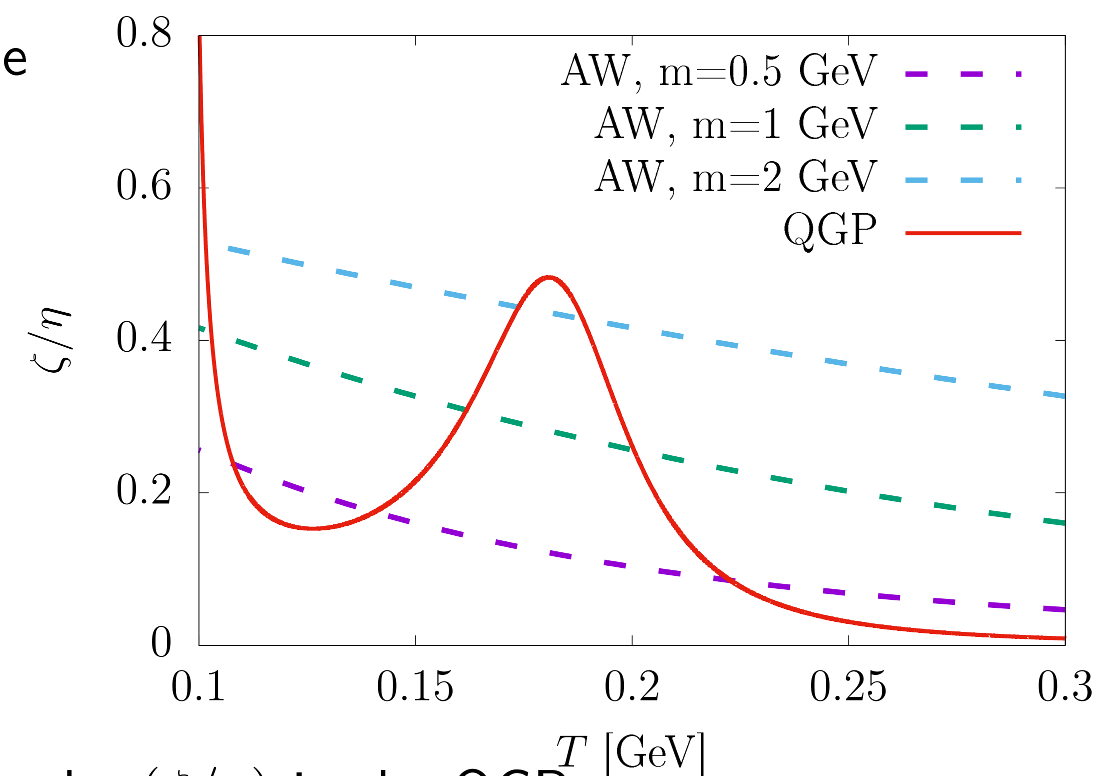
- RTA allows, e.g. η to be specified by setting

$$\tau_R = \frac{\eta}{\alpha_0^{(2)}},$$

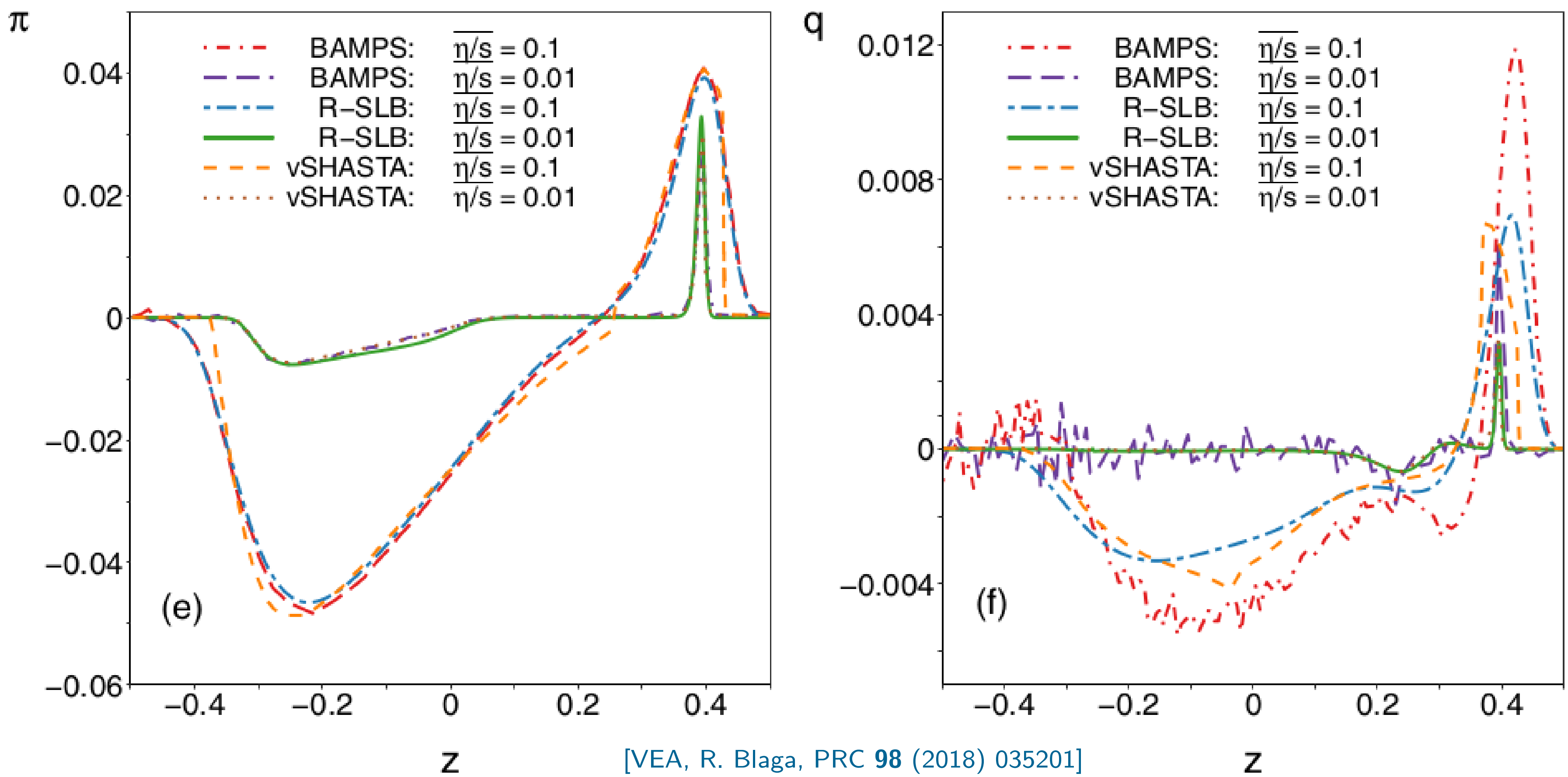
however, ζ/η is fixed uniquely by

$$\frac{\zeta}{\eta} = \frac{m^2\alpha_0^{(0)}}{3\alpha_0^{(2)}},$$

which does not resemble the (ζ/η) in the QGP.



RTA vs BAMPS



[VEA, R. Blaga, PRC **98** (2018) 035201]

- Also for UR hard spheres, $(\kappa T/\eta)_{\rm HS} \simeq 0.125$, whereas $(\kappa T/\eta)_{\rm AW} = 5/48 \simeq 0.104$. DNMR, PRD **85** (2012) 114047
- Fixing η via τ_R gives good agreement with BAMPS for $\pi^{\mu\nu}$ but q^μ is not captured correctly.
- **Aim of this work:** Extend RTA with extra parameters allowing multiple transport coefficients to be controlled independently.

Shakhov-like extension

[Ambruș, Molnár, under review]

- We consider a Shakhov-like extension: [Shakhov, Fluid Dyn. 3 (1968) 112]

$$C_{\rm S}[f] = -\frac{E_{\mathbf{k}}}{\tau_R}(f_{\mathbf{k}} - f_{{\rm S}\mathbf{k}}), \tag{10}$$

where $f_{{\rm S}\mathbf{k}} \to f_{0\mathbf{k}}$ as $\delta f_{\mathbf{k}} = f_{\mathbf{k}} - f_{0\mathbf{k}} \to 0$.

- In the Shakhov model, $f_{\mathbf{k}}$ relaxes towards $f_{0\mathbf{k}}$ on a modified path compared to AW.

- The cons. eqs. $\partial_\mu N^\mu = \partial_\nu T^{\mu\nu} = 0$ imply:

$$u_\mu N^\mu = u_\mu N^\mu_{\rm S}, \quad u_\nu T^{\mu\nu} = u_\nu T^{\mu\nu}_{\rm S}, \tag{11}$$

which allows for plenty of degrees of freedom (δn, $\delta\epsilon$, W^μ, etc).

- For simplicity, we stick to the Landau matching conditions:

$$\delta n = \delta\epsilon = 0, \qquad T^{\mu\nu} u_\nu = \epsilon u^\mu. \tag{12}$$

Shakohv-like extension

- Employing the Chapman-Enskog procedure gives

$$\delta f_{\mathbf{k}} - \delta f_{\mathrm{S}\mathbf{k}} = -\frac{\tau_R}{E_{\mathbf{k}}} k^{\mu} \partial_{\mu} f_{0\mathbf{k}}, \tag{13}$$

leading to

$$\Pi - \Pi_{\mathrm{S}} = -\zeta_R \theta, \quad V^{\mu} - V_{\mathrm{S}}^{\mu} = \kappa_R \nabla^{\mu} \alpha, \quad \pi^{\mu\nu} - \pi_{\mathrm{S}}^{\mu\nu} = 2\eta_R \sigma^{\mu\nu}. \tag{14}$$

- We seek to replace ζ_{R} etc by independent transport coefficients:

$$\begin{aligned} \Pi &\simeq -\zeta_{\mathrm{S}} \theta, & V^{\mu} &\simeq \kappa_{\mathrm{S}} \nabla^{\mu} \alpha, & \pi^{\mu\nu} &\simeq 2\eta_{\mathrm{S}} \sigma^{\mu\nu}, \\ \zeta_{\mathrm{S}} &= \frac{\tau_{\Pi}}{\tau_R} \zeta_R, & \kappa_{\mathrm{S}} &= \frac{\tau_V}{\tau_R} \kappa_R, & \eta_{\mathrm{S}} &= \frac{\tau_{\pi}}{\tau_R} \eta_R. \end{aligned} \tag{15}$$

- Eq. (15) can be obtained from Eq. (14) when

$$\Pi_{\mathrm{S}} = \Pi \left(1 - \frac{\tau_{\Pi}}{\tau_R}\right), \quad V_{\mathrm{S}}^{\mu} = V^{\mu} \left(1 - \frac{\tau_V}{\tau_R}\right), \quad \pi_{\mathrm{S}}^{\mu\nu} = \pi^{\mu\nu} \left(1 - \frac{\tau_{\pi}}{\tau_R}\right). \tag{16}$$

Minimal $\delta f_{\mathrm{S}\mathbf{k}}$

- Writing $f_{\mathrm{S}\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathrm{S}\mathbf{k}}$, we require:

$$\begin{array}{l} \text{Bulk visc. p.} \\ \text{Particle cons.} \\ \text{Energy cons.} \end{array} \Rightarrow \int dK \begin{pmatrix} 1 \\ E_\mathbf{k} \\ E_\mathbf{k}^2 \end{pmatrix} \delta f_{\mathrm{S}\mathbf{k}} \equiv \begin{pmatrix} \rho_{\mathrm{S};0} \\ \rho_{\mathrm{S};1} \\ \rho_{\mathrm{S};2} \end{pmatrix} = \begin{pmatrix} -3\Pi_\mathrm{S}/m^2 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{array}{l} \text{Diff. current} \\ \text{Mom. cons.} \end{array} \Rightarrow \int dK \begin{pmatrix} 1 \\ E_\mathbf{k} \end{pmatrix} k^{\langle\mu\rangle} \delta f_{\mathrm{S}\mathbf{k}} \equiv \begin{pmatrix} \rho^\mu_{\mathrm{S};0} \\ \rho^\mu_{\mathrm{S};1} \end{pmatrix} = \begin{pmatrix} V^\mu_\mathrm{S} \\ 0 \end{pmatrix},$$

$$\text{SS tens.} \Rightarrow \int dK k^{\langle\mu} k^{\nu\rangle} \delta f_{\mathrm{S}\mathbf{k}} \equiv \rho^{\mu\nu}_{\mathrm{S};0} = \pi^{\mu\nu}_\mathrm{S}, \tag{17}$$

with $k^{\langle\mu\rangle} = \Delta^\mu_\alpha k^\alpha$ and $k^{\langle\mu} k^{\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta} k^\alpha k^\beta$ *irreducible* tensors.

- The solution can be written as $\delta f_{\mathrm{S}\mathbf{k}} = f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}} \mathbb{S}_\mathbf{k}$, where

$$\mathbb{S}_\mathbf{k} = -\frac{3\Pi}{m^2}\left(1 - \frac{\tau_R}{\tau_\Pi}\right)\mathcal{H}^{(0)}_{\mathbf{k}0} + k_{\langle\mu\rangle} V^\mu \left(1 - \frac{\tau_R}{\tau_V}\right)\mathcal{H}^{(1)}_{\mathbf{k}0} + k_{\langle\mu} k_{\nu\rangle} \pi^{\mu\nu}\left(1 - \frac{\tau_R}{\tau_\pi}\right)\mathcal{H}^{(2)}_{\mathbf{k}0}. \tag{18}$$

- $\mathcal{H}^{(\ell)}_{\mathbf{k}0}$ are polynomials in $E_\mathbf{k}$ satisfying (17).

[DNMR, PRD **85** (2012) 114047]

- Introducing the thermodynamic integrals,

$$I_{nq} = \frac{1}{(2q+1)!!} \int dK \, E_{\mathbf{k}}^{n-2q} (-\Delta^{\alpha\beta} k_\alpha k_\beta)^q f_{0\mathbf{k}},$$
$$J_{nq} = \frac{1}{(2q+1)!!} \int dK \, E_{\mathbf{k}}^{n-2q} (-\Delta^{\alpha\beta} k_\alpha k_\beta)^q f_{0\mathbf{k}} \tilde{f}_{0\mathbf{k}}, \tag{19}$$

Eqs. (17) can be solved by taking $\mathcal{H}_{0\mathbf{k}}^{(\ell)}$ as polynomials of order $N_\ell = 2 - \ell$:

$$\mathcal{H}_{\mathbf{k}0}^{(0)} = \frac{G_{33} - G_{23} E_{\mathbf{k}} + G_{22} E_{\mathbf{k}}^2}{J_{00} G_{33} - J_{10} G_{23} + J_{20} G_{22}},$$
$$\mathcal{H}_{\mathbf{k}0}^{(1)} = \frac{J_{31} E_{\mathbf{k}} - J_{41}}{J_{21} J_{41} - J_{31}^2}, \qquad \mathcal{H}_{\mathbf{k}0}^{(2)} = \frac{1}{2 J_{42}}, \tag{20}$$

where $G_{nm} = J_{n0} J_{m0} - J_{n-1,0} J_{m+1,0}$.

Entropy production

- The entropy current is given by [classical stat. used for simplicity]

$$S^\mu = -\int dK\, k^\mu \left(f_{\mathbf{k}} \ln f_{\mathbf{k}} - f_{\mathbf{k}} \right). \tag{21}$$

- In the Shakhov model, $k^\mu \partial_\mu f = C_{\rm S}[f]$ and

$$\partial_\mu S^\mu = -\int dK\, C_{\rm S}[f] \ln f_{\mathbf{k}} = \frac{1}{\tau_R} \int dK\, E_{\mathbf{k}} (\delta f_{\mathbf{k}} - \delta f_{{\rm S}\mathbf{k}}) \ln f_{\mathbf{k}}. \tag{22}$$

- $\partial_\mu S^\mu$ difficult for generic $f_{\mathbf{k}}$.
- When $\phi_{\mathbf{k}} = \delta f_{\mathbf{k}} / f_{0\mathbf{k}}$ is small, detailed manipulations lead to

$$\partial_\mu S^\mu \simeq \frac{\beta}{\zeta_{\rm S}} \Pi^2 - \frac{1}{\kappa_{\rm S}} V_\mu V^\mu + \frac{\beta}{2\eta_{\rm S}} \pi_{\mu\nu} \pi^{\mu\nu} \geq 0. \tag{23}$$

- Close to eq., the S-model satisfies the $2^{\rm nd}$ law of thermodynamics.
- Proof far from eq. unavailable even for non-rel. Shakhov!

Application: Bjorken flow

- Bjorken model: flow invariant under longitudinal boosts:

$$u^\mu \partial_\mu = \frac{t}{\tau}\partial_t + \frac{z}{\tau}\partial_z, \quad \tau = \sqrt{t^2 - z^2}, \quad \eta_s = \tanh^{-1}(z/t). \tag{24}$$

- In Bjorken coordinates $(\tau, \mathbf{x}_\perp, \eta_s)$,

$$T^{\mu\nu} = \mathrm{diag}(e, P_T, P_T, \tau^{-2} P_L),$$
$$P_T = P + \Pi - \frac{\pi_d}{2}, \qquad P_L = P + \Pi + \pi_d. \tag{25}$$

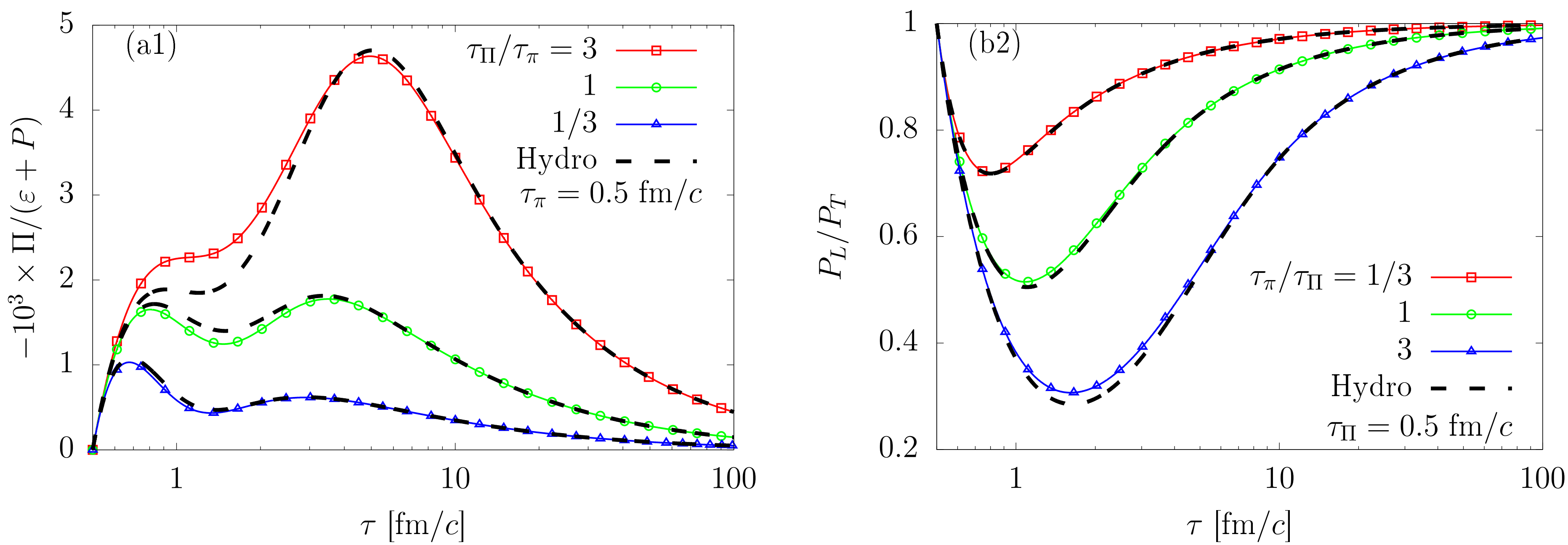
- In 2^{nd}-order hydro, we have: [Denicol, Florkowski, Ryblewski, Strickland, PRC **90** (2014) 044905]

$$\tau\dot{\epsilon} + \epsilon + P_L = 0, \tag{26a}$$

$$\tau\dot{\Pi} + \left(\frac{\delta_{\Pi\Pi}}{\tau_\Pi} + \frac{\tau}{\tau_\Pi}\right)\Pi + \frac{\lambda_{\Pi\pi}}{\tau_\Pi}\pi_d = -\frac{\zeta}{\tau_\Pi},$$
$$\tau\dot{\pi}_d + \left(\frac{\delta_{\pi\pi}}{\tau_\pi} + \frac{\tau_{\pi\pi}}{3\tau_\pi} + \frac{\tau}{\tau_\pi}\right)\pi_d + \frac{2\lambda_{\pi\Pi}}{3\tau_\pi}\Pi = -\frac{4\eta}{3\tau_\pi}. \tag{26b}$$

- We employ the Shakhov model to control ζ independently from η.

Shakhov model: ζ vs. η



- Choosing $\tau_R = \tau_\Pi$, the Shakhov distribution becomes

$$f_{S\mathbf{k}} = f_{0\mathbf{k}} \left[1 + \frac{\beta^2 k_\mu k_\nu \pi^{\mu\nu}}{2(e+P)} \left(1 - \frac{\tau_\Pi}{\tau_\pi} \right) \right]. \tag{27}$$

- Left panel: τ_π is fixed and τ_Π is varied using the Shakhov model.
- Right panel: τ_Π is fixed and τ_π is varied using the Shakhov model.
- $m = 1$ GeV; $\tau_0 = 0.5$ fm; $\beta_0^{-1} = 0.6$ GeV; For $\tau_\pi = 0.5$ fm, $4\pi\eta/s \simeq 3.3$ at $\tau = \tau_0$.

Application: Sound waves

- We now consider an infinitesimal perturbation propagating in an ultrarelativistic fluid at rest.
- Writing $u^\mu \simeq (1,0,0,\delta v)$, $\epsilon = \epsilon_0 + \delta\epsilon$ and $n = n_0 + \delta n$, we have

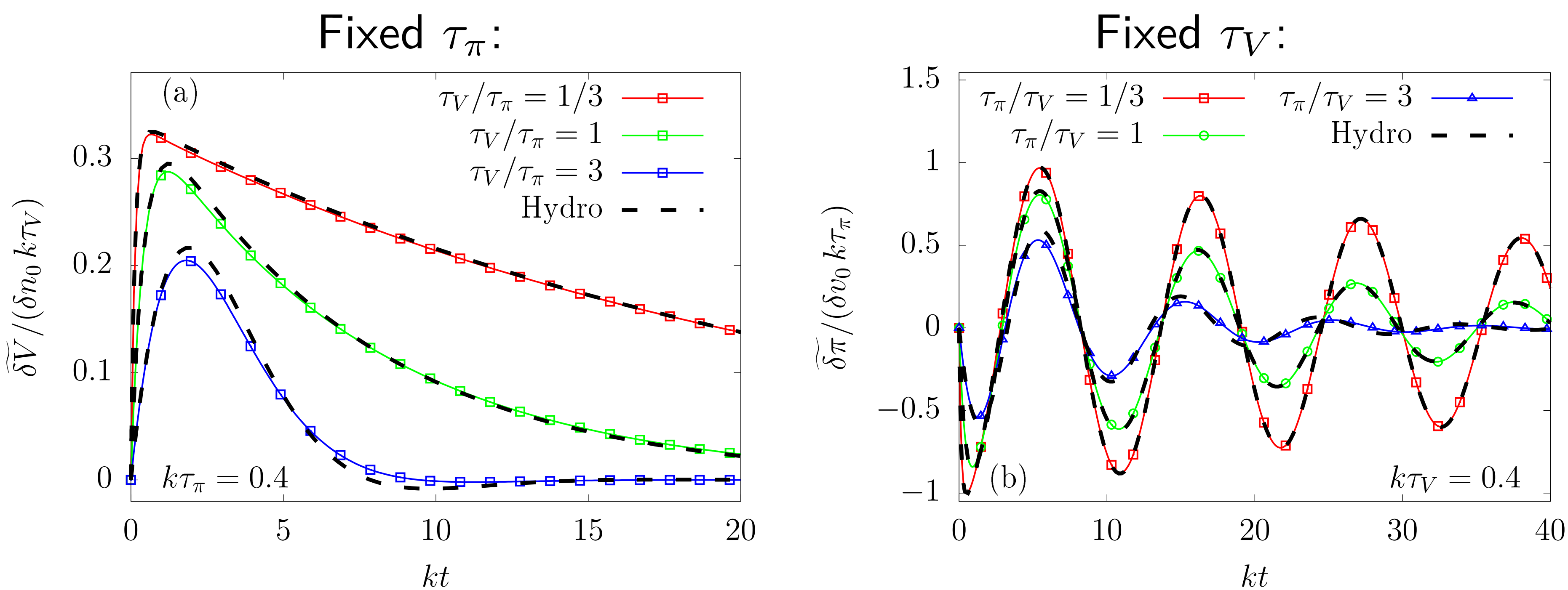
$$\begin{aligned}
\partial_t \delta n + n_0 \partial_z \delta v + \partial_z \delta V &= 0,\\
\partial_t \delta\epsilon + (\epsilon_0 + P_0)\partial_z \delta v &= 0,\\
(\epsilon_0 + P_0)\partial_t \delta v + \partial_z \delta P + \partial_z \delta\pi &= 0,\\
\tau_V \partial_t \delta V + \delta V + \kappa \partial_z \delta\alpha - \ell_{V\pi}\partial_z \delta\pi &= 0,\\
\tau_\pi \partial_t \delta\pi + \delta\pi + \frac{4\eta}{3}\partial_z \delta v + \frac{2}{3}\ell_{\pi V}\partial_z \delta V &= 0,
\end{aligned} \tag{28}$$

  where $\delta V = V^z$ and $\delta\pi = \pi^{zz}/\gamma^2$.
- **In RTA, $\ell_{V\pi} = \ell_{\pi V} = 0$.** [Ambruș, Molnár, Rischke, PRD **106** (2022) 076005]
- We track the time evolution of the amplitudes

$$\widetilde{\delta V} = \frac{2}{L}\int_0^L dz\, \delta V\, \cos(kz), \quad \widetilde{\delta\pi} = \frac{2}{L}\int_0^L dz\, \delta\pi\, \sin(kz). \tag{29}$$

- We employ the Shakhov model to control κ independently from η.

Shakhov model: κ vs. η



- Setting $\tau_R = \tau_\pi$, the Shakhov distribution becomes

$$f_{S\mathbf{k}} = f_{0\mathbf{k}}\left[1 + \frac{k_\mu V^\mu}{P}(\beta E_\mathbf{k} - 5)\left(1 - \frac{\tau_\pi}{\tau_V}\right)\right]. \qquad (30)$$

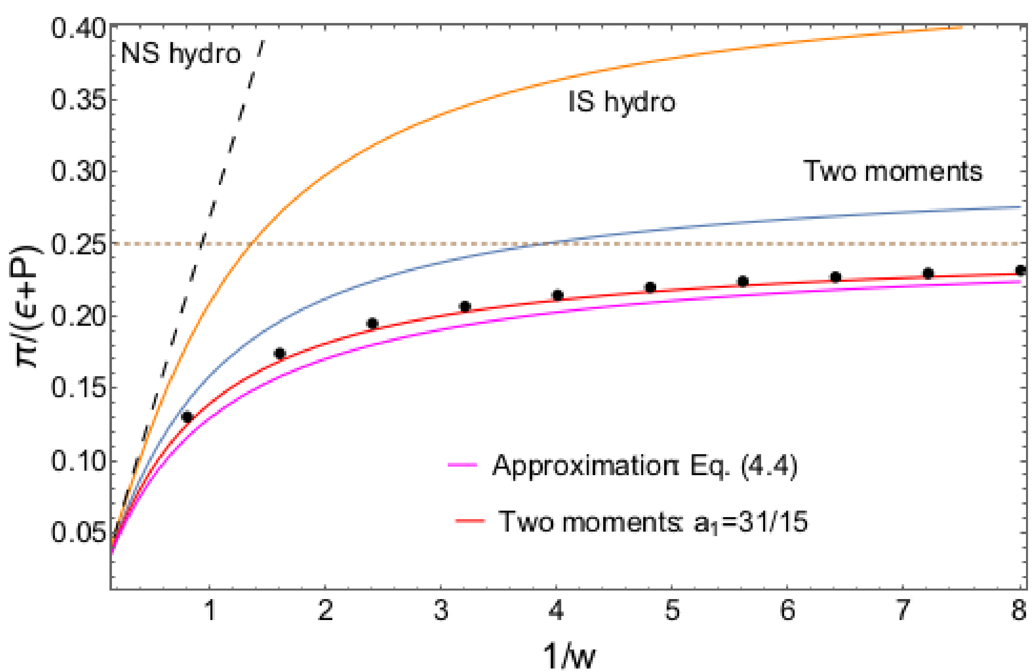
Beyond first order: second-order transport coefficients?

- Relativistic hydrodynamics must obey causality ⇒ first-order theories are excluded.
- One example is the Israel-Stewart-type hydro, by which e.g. $\pi^{\mu\nu}$ evolves according to $\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} + \mathcal{J}^{\mu\nu} + \mathcal{R}^{\mu\nu}$, with

$$\begin{aligned} \mathcal{J}^{\mu\nu} &= 2\tau_\pi \pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} \\ &\quad - \tau_{\pi V} V^{\langle\mu}\dot{u}^{\nu\rangle} + \ell_{\pi V}\nabla^{\langle\mu}V^{\nu\rangle} + \lambda_{\pi V} V^{\langle\mu}\nabla^{\nu\rangle}\alpha, \\ \mathcal{R}^{\mu\nu} &= \varphi_6 \Pi\pi^{\mu\nu} + \varphi_7 \pi^{\lambda\langle\mu}\pi_\lambda^{\nu\rangle} + \varphi_8 V^{\langle\mu}V^{\nu\rangle}. \end{aligned} \tag{31}$$

- In RTA, $\mathcal{R}^{\mu\nu} = 0$.
- 2^{nd}-order t.c. are important e.g. in preeq!
- In conformal RTA, $\delta_{\pi\pi} + \tau_{\pi\pi}/3 = 38/21$.
- Solving hydro with $\delta_{\pi\pi} + \tau_{\pi\pi}/3 = 31/15$ gives much better agreement with RTA!

[J.-P. Blaizot, L. Yan, PRC **104** (2021) 055201]



- Etc...

Second-order hydro from KT

- In the method of moments, second-order hydro can be derived using:
  - Irreducible moments of $\delta f_{\mathbf{k}}$: $\rho_r^{\mu_1\cdots\mu_\ell} = \int dK E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle}\delta f_{\mathbf{k}}$.
  - Irreducible moments of $C[f]$: $C_r^{\mu_1\cdots\mu_\ell} = \int dK E_{\mathbf{k}}^r k^{\langle\mu_1}\cdots k^{\mu_\ell\rangle} C[f]$.
  - Define collision matrix via $C_{r-1}^{\mu_1\cdots\mu_\ell} = -\sum_n \mathcal{A}_{rn}^{(\ell)} \rho_n^{\mu_1\cdots\mu_\ell}$.
  - Define inverse matrix $\tau_{rn}^{(\ell)}$ via $\sum_n \tau_{rn}^{(\ell)} \mathcal{A}_{nm}^{(\ell)} = \delta_{rm}$.
- For example, the first-order transport coeffs. are

$$\zeta_r = \frac{m^2}{3}\sum_n \tau_{rn}^{(0)}\alpha_n^{(0)}, \quad \kappa_r = \sum_n \tau_{rn}^{(1)}\alpha_n^{(1)}, \quad \eta_r = \sum_n \tau_{rn}^{(2)}\alpha_n^{(2)}.$$

- The relaxation times can be obtained via

$$\tau_\Pi = \sum_n \tau_{0n}^{(0)}\mathcal{C}_n^{(0)}, \quad \tau_V = \sum_n \tau_{0n}^{(1)}\mathcal{C}_n^{(1)}, \quad \tau_\pi = \sum_n \tau_{0n}^{(2)}\mathcal{C}_n^{(2)}, \qquad (32)$$

  with $\mathcal{C}_n^{(0)} = \zeta_n/\zeta_0$, $\mathcal{C}_n^{(1)} = \kappa_n/\kappa_0$ and $\mathcal{C}_n^{(2)} = \eta_n/\eta_0$.
- ...all other 2nd-order t.c. are computed using $\tau_{0n}^{(\ell)}$ and $\mathcal{C}_n^{(\ell)}$.
- Idea: Use Shakhov model to "manipulate" $\mathcal{A}_{rn}^{(\ell)}$.

From RTA to Shakhov

- In RTA, $C[f] = -\frac{E_{\mathbf{k}}}{\tau_R}\delta f_{\mathbf{k}}$ and [Ambruș, Molnár, Rischke, PRD **106** (2022) 076005]

$$C_{r-1}^{\mu_1\cdots\mu_\ell} = -\frac{1}{\tau_R}\rho_r^{\mu_1\cdots\mu_\ell} \Rightarrow \mathcal{A}_{rn}^{(\ell)} = \frac{\delta_{rn}}{\tau_R} \Rightarrow \tau_{rn}^{(\ell)} = \tau_R\delta_{rn}. \tag{33}$$

- In the Shakhov model, $C_{\rm S} = -\frac{E_{\mathbf{k}}}{\tau_R}[\delta f_{\mathbf{k}} - \delta f_{{\rm S}\mathbf{k}}]$ and

$$C_{r-1}^{\mu_1\cdots\mu_\ell} = -\frac{1}{\tau_R}[\rho_r^{\mu_1\cdots\mu_\ell} - \rho_{{\rm S};r}^{\mu_1\cdots\mu_\ell}], \tag{34}$$

  where $\rho_{{\rm S};r}^{\mu_1\cdots\mu_\ell}$ are essentially arbitrary.
- Imposing $C_{r-1}^{\mu_1\cdots\mu_\ell} = -\sum_n \mathcal{A}_{rn}^{(\ell)}\rho_n^{\mu_1\cdots\mu_\ell}$ suggests taking

$$\rho_{{\rm S};r}^{\mu_1\cdots\mu_\ell} = \sum_n [\delta_{rn} - \tau_R\mathcal{A}_{rn}^{(\ell)}]\rho_n^{\mu_1\cdots\mu_\ell}, \tag{35}$$

  where $\mathcal{A}_{rn}^{(\ell)}$ is the desired collision matrix and $\rho_n^{\mu_1\cdots\mu_\ell}$ is extracted from $f_{\mathbf{k}}$.
- Problem: For a generic $C[f]$, $\mathcal{A}_{rn}^{(\ell)}$ is infinite!

Constructing $\mathbb{S}_{\mathbf{k}}$

[VEA, D. Wagner, arXiv:2401.04017]

- Our approach is to fix a subset of $\rho^{\mu_1\cdots\mu_\ell}_{\mathrm{S};r}$ with:

$$0 \leq \ell \leq L = 2, \qquad -s_\ell \leq r \leq N_\ell, \tag{36}$$

where $s_\ell \equiv$ "shift" and $N_\ell \geq \{2, 1, 0\}$. [Ambruș, Molnár, Rischke, PRD **106** (2022) 076005]

- This can be achieved using the Method of Moments for $\delta f_{\mathrm{S}\mathbf{k}} \equiv f_{\mathrm{S}\mathbf{k}} - f_{0\mathbf{k}} = f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}\mathbb{S}_{\mathbf{k}}$, by setting:

$$\mathbb{S}_{\mathbf{k}} = \sum_{\ell=0}^{L} \sum_{n=-s_\ell}^{N_\ell} \rho^{\mu_1\cdots\mu_\ell}_{\mathrm{S};n} E_{\mathbf{k}}^{-s_\ell} k_{\langle\mu_1} \cdots k_{\mu_\ell\rangle} \widetilde{\mathcal{H}}^{(\ell)}_{\mathbf{k},n+s_\ell}, \tag{37}$$

with $\widetilde{\mathcal{H}}^{(\ell)}_{\mathbf{k}n}$ to be determined.

- Inverting the logic, $\rho^{\mu_1\cdots\mu_\ell}_{\mathrm{S};r}$ are obtained from $\delta f_{\mathrm{S}\mathbf{k}}$ through

$$\rho^{\mu_1\cdots\mu_\ell}_{\mathrm{S};r} = \sum_{n=-s_\ell}^{N_\ell} \rho^{\mu_1\cdots\mu_\ell}_{\mathrm{S};n} \widetilde{\mathcal{F}}^{(\ell)}_{-(r+s_\ell),n+s_\ell},$$

$$\widetilde{\mathcal{F}}^{(\ell)}_{rn} \equiv \frac{\ell!}{(2\ell+1)!!} \int dK\, f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}} E_{\mathbf{k}}^{-2s_\ell - r} (\Delta^{\alpha\beta} k_\alpha k_\beta)^\ell \widetilde{\mathcal{H}}^{(\ell)}_{\mathbf{k}n}. \tag{38}$$

- Imposing $\widetilde{F}^{(\ell)}_{-r,n} = \delta_{rn}$ for $-s_\ell \leq r, n \leq N_\ell$ ensures compatibility with Eq. (21) and fully determines $\widetilde{\mathcal{H}}^{(\ell)}_{\mathbf{k}n}$.

Shakhov collision matrix

- Eq. (38) $\Rightarrow \rho^{\mu_1\cdots\mu_\ell}_{\mathrm{S};r} \neq 0$ even when $r < -s_\ell$ and $r > N_\ell$.
- $\Rightarrow \mathcal{A}^{(\ell)}_{\mathrm{S};rn}$ contains non-trivial entries when $r < -s_\ell$ and $r > N_\ell$:

$$\mathcal{A}^{(\ell)}_{rn} = \begin{pmatrix} \frac{1}{\tau_R}\delta_{rn} & \mathcal{A}^{(\ell)}_{<;rn} & 0 \\ 0 & \mathcal{A}^{(\ell)}_{\mathrm{S};rn} & 0 \\ 0 & \mathcal{A}^{(\ell)}_{>;rn} & \frac{1}{\tau_R}\delta_{rn} \end{pmatrix}, \tag{39}$$

where $\mathcal{A}^{(\ell)}_{</>;rn}$ correspond to $r < -s_\ell$ and $r > N_\ell$, respectively.

- These entries supplement the $\tau_R^{-1}\delta_{rn}$ structure of AW with

$$\mathcal{A}^{(\ell)}_{</>;rn} = -\frac{1}{\tau_R}\widetilde{\mathcal{F}}^{(\ell)}_{-(r+s_\ell),n+s_\ell} + \sum_{j=-s_\ell}^{N_\ell} \widetilde{\mathcal{F}}^{(\ell)}_{-(r+s_\ell),j+s_\ell}\mathcal{A}^{(\ell)}_{\mathrm{S};jn}. \tag{40}$$

Inverse collision matrix

- The inverse matrix $\tau^{(\ell)}_{rn}$ reads

$$\tau^{(\ell)}_{rn} = \begin{pmatrix} \tau_R \delta_{rn} & \tau^{(\ell)}_{<;rn} & 0 \\ 0 & \tau^{(\ell)}_{\mathrm{S};rn} & 0 \\ 0 & \tau^{(\ell)}_{>;rn} & \tau_R \delta_{rn} \end{pmatrix} , \qquad (41)$$

with $\tau^{(\ell)}_{\mathrm{S};rn} = [\mathcal{A}^{(\ell)}_{\mathrm{S};rn}]^{-1}$ a finite $(N_\ell + s_\ell + 1)^2$ matrix and

$$\tau^{(\ell)}_{<,>;rn} = -\tau_R \widetilde{\mathcal{F}}^{(\ell)}_{-(r+s_\ell),n+s_\ell} + \sum_{j=-s_\ell}^{N_\ell} \widetilde{\mathcal{F}}^{(\ell)}_{-(r+s_\ell),j+s_\ell} \tau^{(\ell)}_{\mathrm{S};jn} . \qquad (42)$$

- For example, the shear viscosities $\eta_r = \sum_n \tau^{(2)}_{rn} \alpha^{(2)}_n$ are

$$\eta_{-s_\ell \le r \le N_\ell} = \sum_{n=-s_2}^{N_2} \tau^{(2)}_{\mathrm{S};rn} \alpha^{(2)}_n ,$$

$$\eta_{r,</>} = \tau_R \alpha^{(2)}_r + \sum_{n=-s_2}^{N_2} \widetilde{\mathcal{F}}^{(2)}_{-r-s_2,n+s_2} (\eta_n - \tau_R \alpha^{(2)}_n) . \qquad (43)$$

Tunable coefficients in the Shakhov model

- The t.c. depend on

$$\begin{aligned}
\tau^{(0)}_{0,n\neq 1,2} &: N_0 + s_0 - 1 \text{ entries}; & \mathcal{C}^{(0)}_{n\neq 1,2} &\equiv \frac{\zeta_n}{\zeta_0} : N_0 + s_0 - 2 \text{ extra lines}, \\
\tau^{(1)}_{0,n\neq 1} &: N_1 + s_1 \text{ entries}; & \mathcal{C}^{(1)}_{n\neq 1} &\equiv \frac{\kappa_n}{\kappa_0} : N_1 + s_1 - 1 \text{ extra lines}, \\
\tau^{(2)}_{0n} &: N_2 + s_2 + 1 \text{ entries}; & \mathcal{C}^{(2)}_{n} &\equiv \frac{\eta_n}{\eta_0} : N_2 + s_2 \text{ extra lines},
\end{aligned} \tag{44}$$

so in total:

$$2(N_0 + s_0 + N_1 + s_1 + N_2 + s_2) - 3 \,\text{transport coefficients}, \tag{45}$$

plus a hidden degree of freedom given by τ_R.

- For an ultrarelativistic gas, the scalar sector is not important, leaving in total

$$2(N_1 + s_1 + N_2 + s_2) \text{ transport coefficients}, \tag{46}$$

plus τ_R.

Example: shear-diffusion coupling

- Consider a longitudinal wave propagating along z.
- The linearized hydro equations for $\delta\pi \equiv \pi^{zz}$ and $\delta V \equiv V^z$ read

$$\tau_V \partial_t \delta V + \delta V = -\kappa \partial_z \delta\alpha + \ell_{V\pi} \partial_z \delta\pi,$$
$$\tau_\pi \partial_t \delta\pi + \delta\pi = -\frac{4\eta}{3} \partial_z \delta v - \frac{2}{3} \ell_{\pi V} \partial_z \delta V, \tag{47}$$

where the cross couplings read (for an UR classical gas):

$$\ell_{V\pi} = \sum_{r\neq 1} \tau_{0r}^{(1)} \left(\frac{\beta J_{r+2,1}}{\epsilon + P} - \mathcal{C}_{r-1}^{(2)} \right), \quad \ell_{\pi V} = \frac{2}{5} \sum_r \tau_{0r}^{(2)} \mathcal{C}_{r+1}^{(1)}. \tag{48}$$

- In RTA, $\ell_{V\pi} = \tau_R \left(\frac{\beta J_{21}}{\epsilon + P} - \mathcal{C}_{-1}^{(2)} \right)$ and $\ell_{\pi V} = \tau_R \mathcal{C}_1^{(1)}$ both vanish:

$$J_{21} = nT = \frac{1}{3}\epsilon, \qquad \mathcal{C}_{-1}^{(2)} = \frac{\alpha_{-1}^{(2)}}{\alpha_0^{(2)}} = \frac{\beta}{4} \qquad \Rightarrow \qquad \ell_{V\pi} = 0,$$
$$\kappa_1 = \alpha_1^{(1)} = 0, \qquad \mathcal{C}_1^{(1)} = \frac{\alpha_1^{(1)}}{\alpha_0^{(1)}} = 0 \qquad \Rightarrow \qquad \ell_{\pi V} = 0. \tag{49}$$

- We aim to control independently 4 t.c.: κ, η, $\ell_{V\pi}$ and $\ell_{\pi V}$.

Example: shear-bulk coupling

- To illustrate the capabilities of the Shakhov model in the case of finite m, we consider again the Bjorken flow problem.
- In MIS hydro, the diffusive quantities evolve according to

$$\tau_\Pi \frac{d\Pi}{d\tau} + \Pi = -\frac{1}{\tau}\left(\zeta + \delta_{\Pi\Pi}\Pi + \lambda_{\Pi\pi}\pi_d\right) ,$$
$$\tau_\pi \frac{d\pi_d}{d\tau} + \pi_d = -\frac{1}{\tau}\left[\frac{4\eta}{3} + \left(\delta_{\pi\pi} + \frac{\tau_{\pi\pi}}{3}\right)\pi_d + \frac{2\lambda_{\pi\Pi}}{3}\Pi\right]. \quad (50)$$

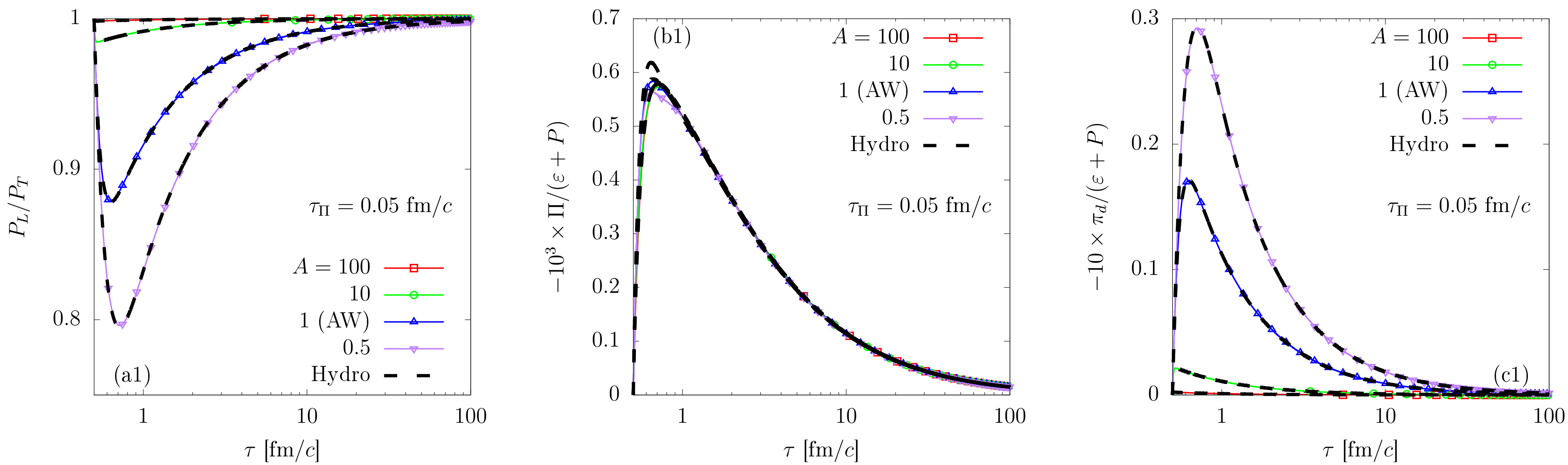
- Our aim is to separately tune ζ, η and $\lambda_{\Pi\pi}$, i.e.

$$\frac{\lambda_{\Pi\pi}}{\tau_\Pi} = A\frac{\lambda^R_{\Pi\pi}}{\tau_R}, \quad \eta = H\eta_R, \quad \zeta = \zeta_R, \quad (51)$$

  where $\lambda^R_{\Pi\pi} = \frac{m^2}{3}\tau_R\left(\mathcal{R}^{(2)}_{-2} + \frac{J_{10}}{J_{30}}\right)$ is the RTA expression, while A and H are arbitrary functions.
- This can be achieved using the following collision matrix:

$$\mathcal{A}^{(2)}_{\rm S} = \frac{1}{\tau_R H}\begin{pmatrix} 1 & (1-A)\left(\mathcal{R}^{(2)}_{-2} + \frac{J_{10}}{J_{30}}\right) \\ 0 & 1 \end{pmatrix} , \quad \mathcal{R}^{(2)}_{-2} = \frac{\alpha^{(2)}_{-2}}{\alpha^{(2)}_0}. \quad (52)$$

Example: shear-bulk coupling



- For definiteness, we consider $AH = 1 \Rightarrow$ bulk response $\lambda_{\Pi\pi}\pi_d$ remains unchanged (see central panel).
- The Shakhov $f_{S\mathbf{k}} = f_{0\mathbf{k}}\tilde{f}_{0\mathbf{k}}\mathbb{S}_\mathbf{k}$ has $\tilde{f}_{0\mathbf{k}} = 1$ (classical gas) and

$$\mathbb{S}_\mathbf{k} = \left(\pi_{S;-2}\mathfrak{h}^{(2)}_{\mathbf{k}0} + \pi_{S;0}\mathfrak{h}^{(2)}_{\mathbf{k}2}\right)\left(\frac{k_\eta^2}{\tau^2 k_\tau^2} - \frac{k_\perp^2}{2k_\tau^2}\right),$$

$$\pi_{S;r} = \pi_r - \tau_R \mathcal{A}^{(2)}_{S;rn}\pi_n, \quad k^\tau = \frac{tk^t - zk^z}{\tau}, \quad k^\eta = \frac{tk^z - zk^t}{\tau^2},$$

$$\mathfrak{h}^{(2)}_{\mathbf{k}0} = \frac{J_{42} - J_{22}E_\mathbf{k}^2}{2(J_{02}J_{42} - J_{22}^2)}, \quad \mathfrak{h}^{(2)}_{\mathbf{k}2} = \frac{-J_{22} + J_{02}E_\mathbf{k}^2}{2(J_{02}J_{42} - J_{22}^2)}. \qquad (53)$$

Example: shear-diffusion coupling

- We now revisit the longitudinal waves problem and employ the Shakhov model to impose $\ell_{V\pi}$, $\ell_{\pi V} \neq 0$.
- For this purpose, we use the $(N_1, N_2, s_1, s_2) = (1, 0, 0, 1)$ having

$$2(N_1 + s_1 + N_2 + s_2) = 4 \text{ degrees of freedom}, \tag{54}$$

allowing to fix κ, η, $\ell_{V\pi}$ and $\ell_{\pi V}$.

- We take $\mathcal{A}_{\rm S}^{(1)} = 1/\tau_V$ with $\tau_V = 12\kappa/\beta P$.
- Introducing the notation

$$H = \frac{5\eta}{4\tau_\pi P}, \quad L_{V\pi} = \frac{4\ell_{V\pi}}{\beta\tau_V}, \quad L_{\pi V} = \frac{5\beta\ell_{\pi V}}{8\tau_\pi}, \tag{55}$$

we have the constraint $H = 1 + L_{V\pi}L_{\pi V}$, i.e.

$$\tau_\pi = \frac{\tau_R}{1 + L_{V\pi}L_{\pi V}}, \tag{56}$$

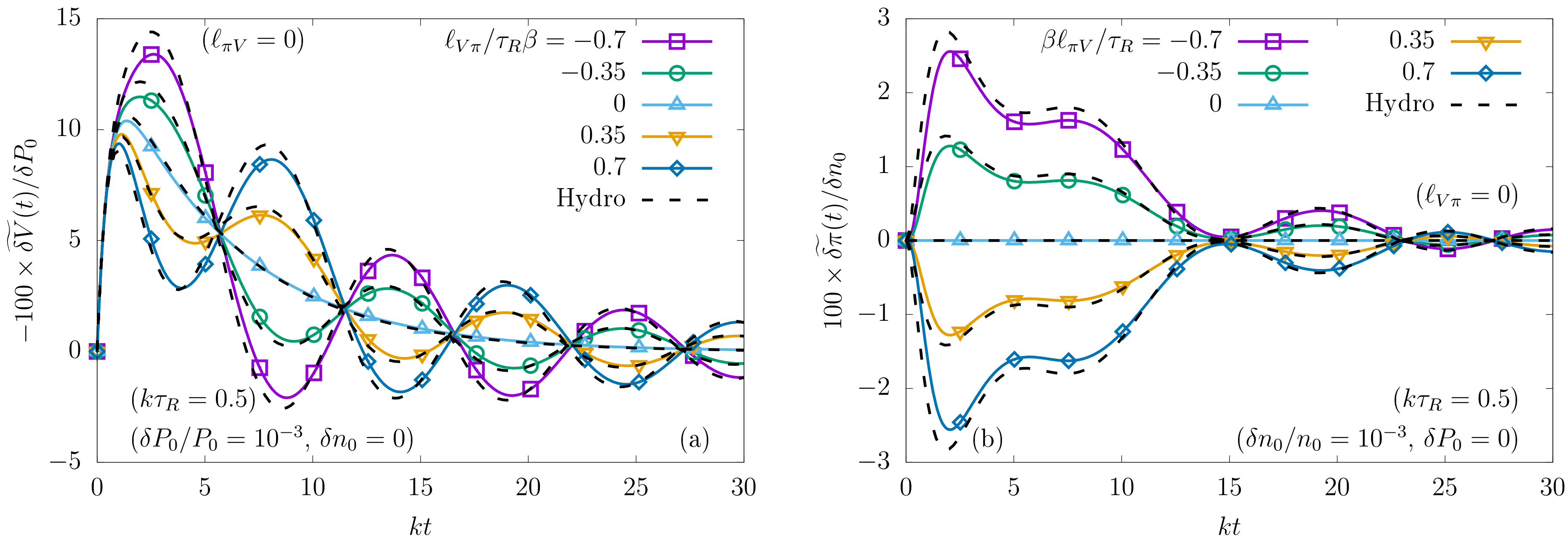
where we take $\tau_R = 5\eta/4P$.

- Then, the matrix reads:

$$\mathcal{A}_{\rm S}^{(2)} = \frac{1-\alpha}{\alpha H \tau_\pi (1-\alpha H)} \begin{pmatrix} H - L_{\pi V} & -\frac{\beta}{4} x \\ -\frac{4}{\beta} L_{\pi V} & H(1 - L_{V\pi}) - x \end{pmatrix}, \tag{57}$$

with $x = H(1 - \alpha - L_{V\pi}) - L_{\pi V} - \frac{1-H}{1-\alpha}$ and $\alpha = 1/2$.

Example: shear-diffusion coupling



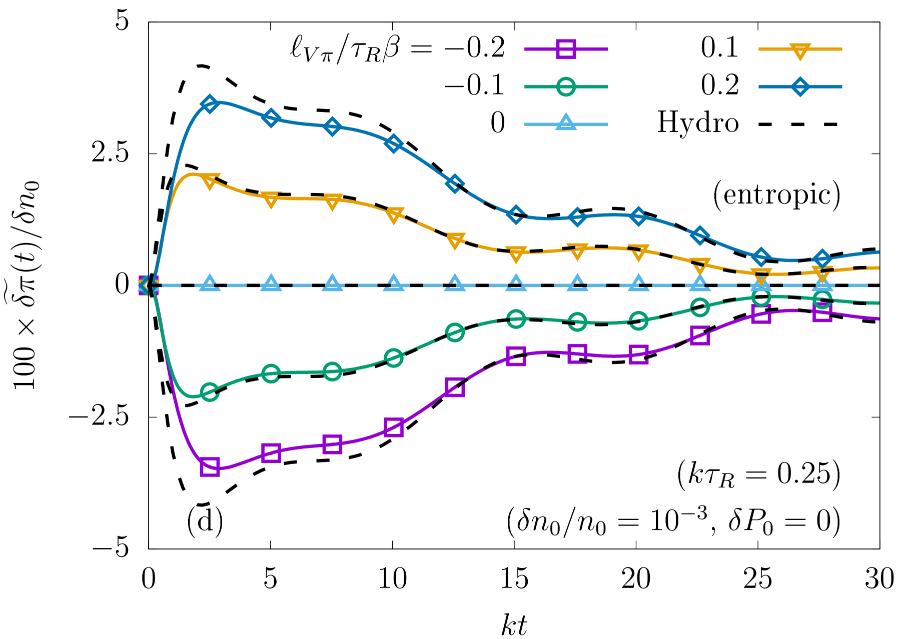
- We first consider $\ell_{\pi V} = 0$ (left panel) and $\ell_{\pi V} = 0$ (right panel):

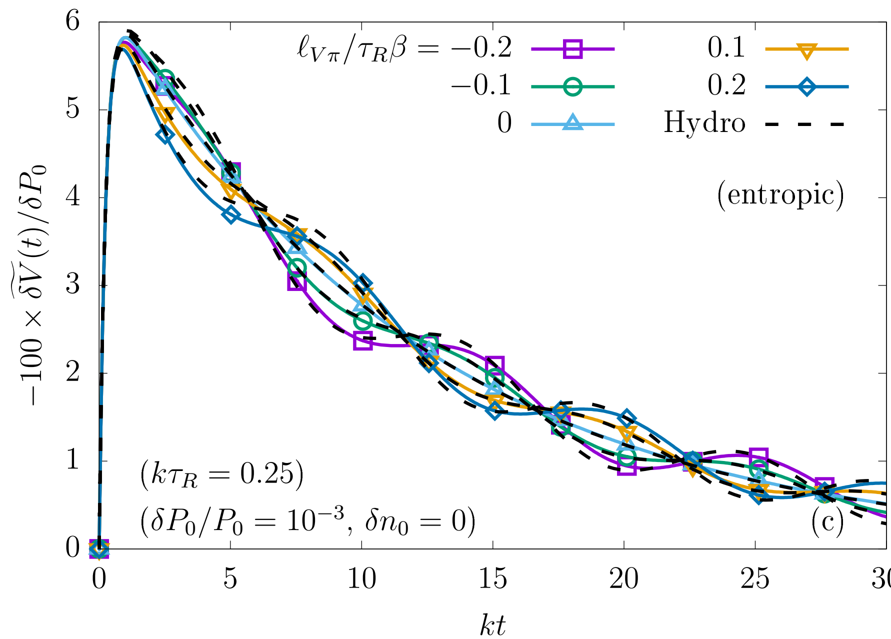
$$\ell_{\pi V} = 0: \qquad\qquad \ell_{V\pi} = 0:$$

$$\mathcal{A}_{\rm S}^{(2)} = \frac{1}{\tau_\pi}\begin{pmatrix} 2 & -\frac{\beta}{4}(1 - 2L_{V\pi}) \\ 0 & 1 \end{pmatrix}, \quad \mathcal{A}_{\rm S}^{(2)} = \frac{2}{\tau_\pi}\begin{pmatrix} 1 - L_{\pi V} & -\beta(\frac{1}{2} - L_{\pi V}) \\ -4\beta L_{\pi V} & \frac{1}{2} + L_{\pi V} \end{pmatrix}. \tag{58}$$

- Very good agreement with hydro observed!

Example: shear-diffusion coupling





- The requirement $\partial_\mu S^\mu \geq 0$ imposes

$$\frac{\ell_{V\pi}}{\kappa} + \frac{\ell_{\pi V}}{2\eta T} = 0 \quad \Rightarrow \quad L_{\pi V} = -3HL_{V\pi}. \tag{59}$$

- In this case, the Shakhov matrix reads:

$$\mathcal{A}_{\rm S}^{(2)} = \frac{2}{\tau_\pi(2-H)} \begin{pmatrix} 1 + 3L_{V\pi} & \frac{\beta}{8}(12L_{V\pi}^2 - 4L_{V\pi} - 1) \\ \frac{12}{\beta} L_{V\pi} & 6L_{V\pi}^2 - 3L_{V\pi} + \frac{1}{2} \end{pmatrix}, \tag{60}$$

- Again, very good agreement with hydro observed!

Ultrarelativistic hard spheres (URHS)

- The t.c. of the URHS model are: [D. Wagner, A. Palermo, VEA, PRD **106** (2022) 016013]

[D. Wagner, VEA, E. Molnár, arXiv: 2309.09335]

$\kappa\sigma$	$\tau_V\,[\lambda_{\rm mfp}]$	$\delta_{VV}\,[\tau_V]$	$\ell_{V\pi}\,[\tau_V] = \tau_{V\pi}\,[\tau_V]$	$\lambda_{VV}\,[\tau_V]$	$\lambda_{V\pi}\,[\tau_V]$
0.15892	2.0838	1	0.028371β	0.89862	0.069273β

$\eta\sigma\beta$	$\tau_\pi\,[\lambda_{\rm mfp}]$	$\delta_{\pi\pi}\,[\tau_\pi]$	$\ell_{\pi V}\,[\tau_\pi]$	$\tau_{\pi V}\,[\tau_\pi]$	$\tau_{\pi\pi}\,[\tau_\pi]$	$\lambda_{\pi V}\,[\tau_\pi]$
1.2676	1.6557	$4/3$	$-0.56960/\beta$	$-2.2784/\beta$	1.6945	$0.20503/\beta$

- The t.c. of RTA with $\eta_R = \eta_{\rm HS}$ are

$\kappa\sigma$	$\tau_V\,[\lambda_{\rm mfp}]$	$\delta_{VV}\,[\tau_V]$	$\ell_{V\pi}\,[\tau_V] = \tau_{V\pi}\,[\tau_V]$	$\lambda_{VV}\,[\tau_V]$	$\lambda_{V\pi}\,[\tau_V]$
0.13204	1.5845	1	0	$3/5$	$\beta/16$

$\eta\sigma\beta$	$\tau_\pi\,[\lambda_{\rm mfp}]$	$\delta_{\pi\pi}\,[\tau_\pi]$	$\ell_{\pi V}\,[\tau_\pi]$	$\tau_{\pi V}\,[\tau_\pi]$	$\tau_{\pi\pi}\,[\tau_\pi]$	$\lambda_{\pi V}\,[\tau_\pi]$
1.2676	1.5845	$4/3$	0	0	$10/7$	0

- RTA-HS mismatch for almost all coefficients, except $\delta_{VV} = \tau_V$ and $\delta_{\pi\pi} = 4\tau_\pi/3$, which are fixed for an UR gas.
- To align all transport coefficients, we need 11 parameters!

Various (N_1, N_2, s_1, s_2) models

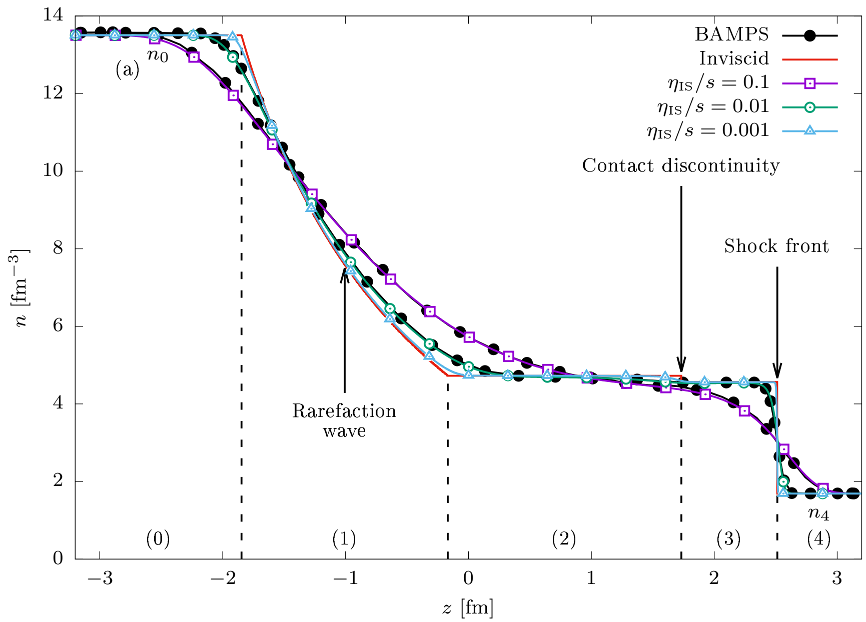
- A Shakhov model with parameters (N_1, N_2, s_1, s_2) provides $2(N_1 + N_2 + s_1 + s_2)$.
- To test the effect of various t.c., we employed several models:
- AW: τ_R is used to fix $\eta_R = \eta_{\rm HS}$.
- (1000): Fixes η and κ.
- (1001): discussed previously, fixes $(\kappa, \eta, \ell_{V\pi}, \ell_{\pi V})$.
- (1012): has $2 \times 4 = 8$ free entries and fixes everything except λ_{VV} and $\lambda_{V\pi}$.
- (2102): has $2 \times 5 = 10$ free entries and fixes everything.

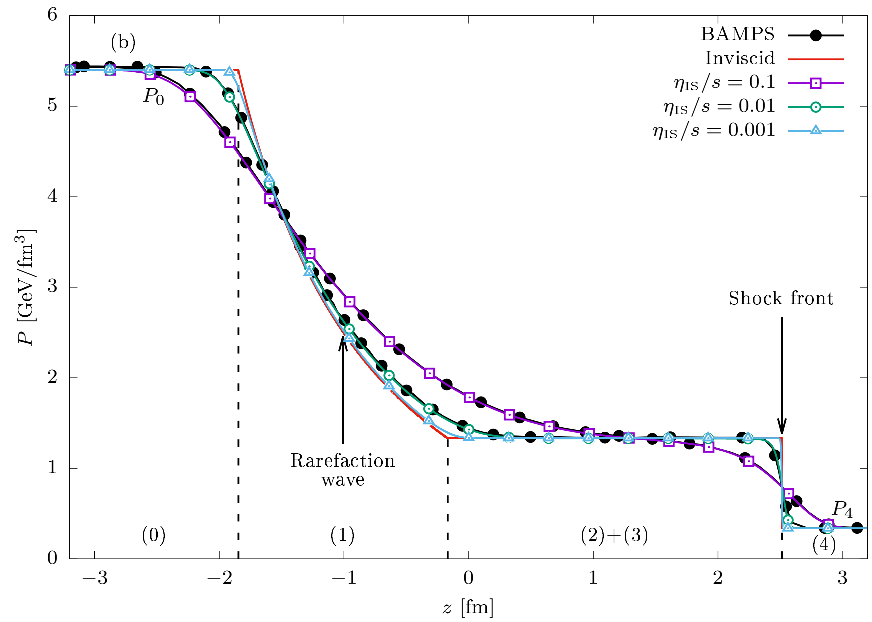
Models used

Model	$\eta\sigma\beta$	$\tau_\pi/\lambda_{\rm mfp}$	$\ell_{\pi V}/\tau_\pi$	$\tau_{\pi\pi}/\tau_\pi$	$\beta\lambda_{\pi V}/\tau_\pi$
HS	**1.2676**	**1.6557**	**−0.56960**	**1.6945**	**0.20503**
AW	**1.2676**	1.5845	0	1.4286	0
1000	**1.2676**	1.5845	0	1.4286	0
1001	**1.2676**	1.6457	**−0.56960**	1.7607	0
1012	**1.2676**	**1.6557**	**−0.56960**	**1.6945**	**0.20503**
2012	**1.2676**	**1.6557**	**−0.56960**	**1.6945**	**0.20503**

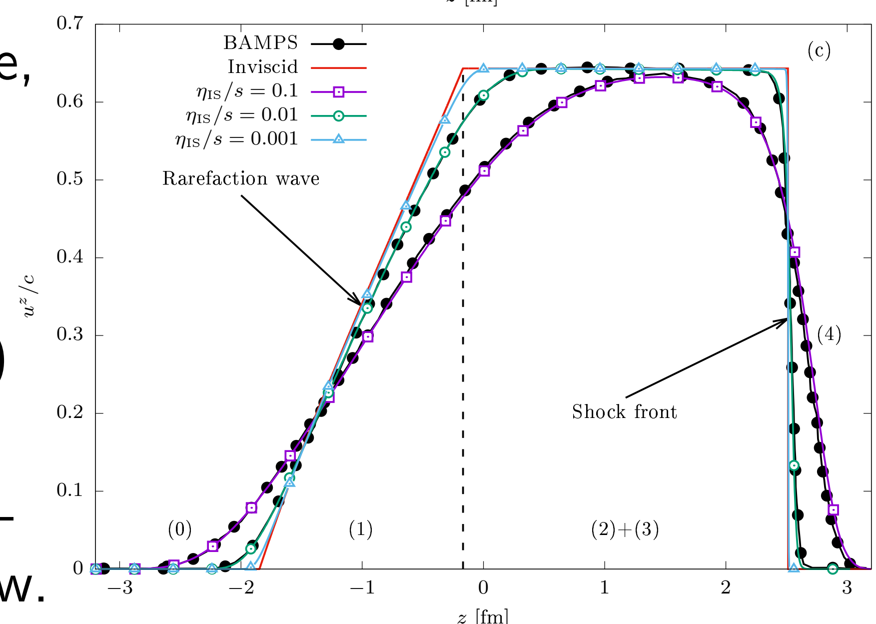
Model	$\kappa\sigma$	$\tau_V/\lambda_{\rm mfp}$	$\ell_{V\pi}/\beta\tau_V$	λ_{VV}/τ_V	$\lambda_{V\pi}/\beta\tau_V$
HS	**0.15892**	**2.0838**	**0.028371**	**0.89862**	**0.069273**
AW	0.13204	1.5845	0	0.6	0.0625
1000	**0.15892**	1.5845	0	0.6	0.0625
1001	**0.15892**	1.9070	**0.028371**	0.6	0.055407
1012	**0.13204**	**2.0838**	**0.028371**	0.762023	0.062933
2012	**0.15892**	**2.0838**	**0.028371**	**0.89862**	**0.069273**

Sod shock tube: convergence properties



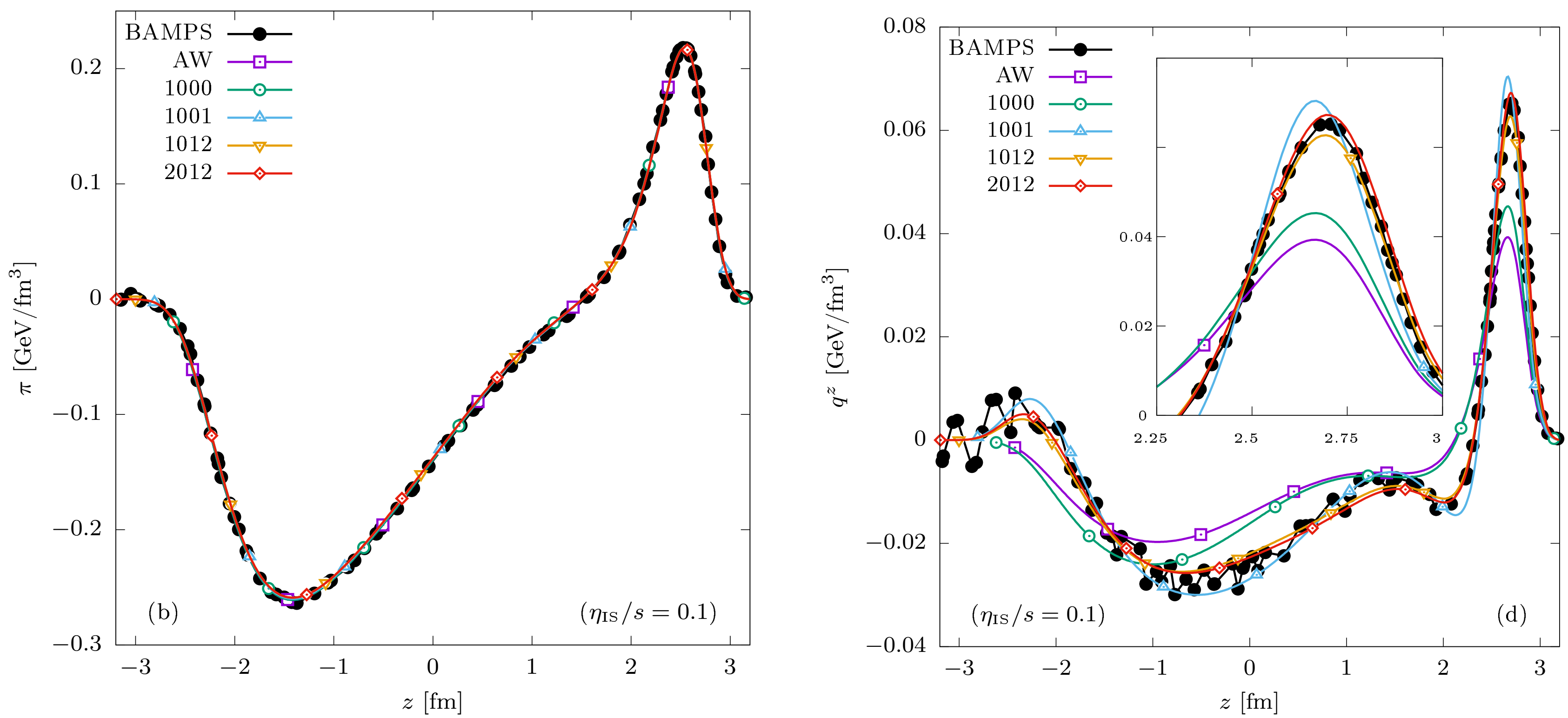


- To validate the numerical scheme, we compared AW results to BAMPS for various fixed η/s.
- As $\eta/s \to 0$, our results approach the inviscid (analytical) solution.
- AW and all Shakhov implementations are in excellent agreement w. BAMPS for the eq. quantits. (n, P, u).



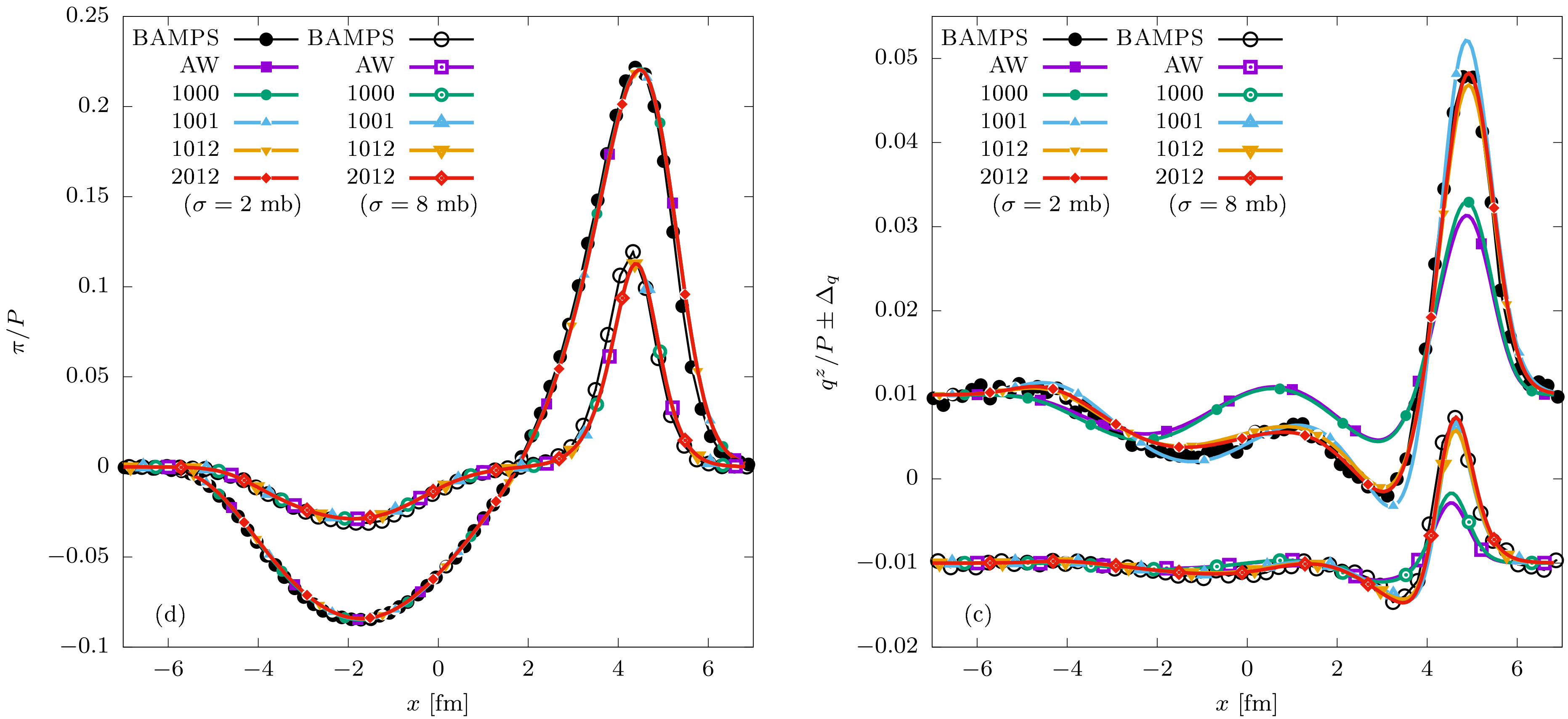
Sod shock tube: Comparison to BAMPS

[Bouras *et al*, PRC **82** (2010) 024910]



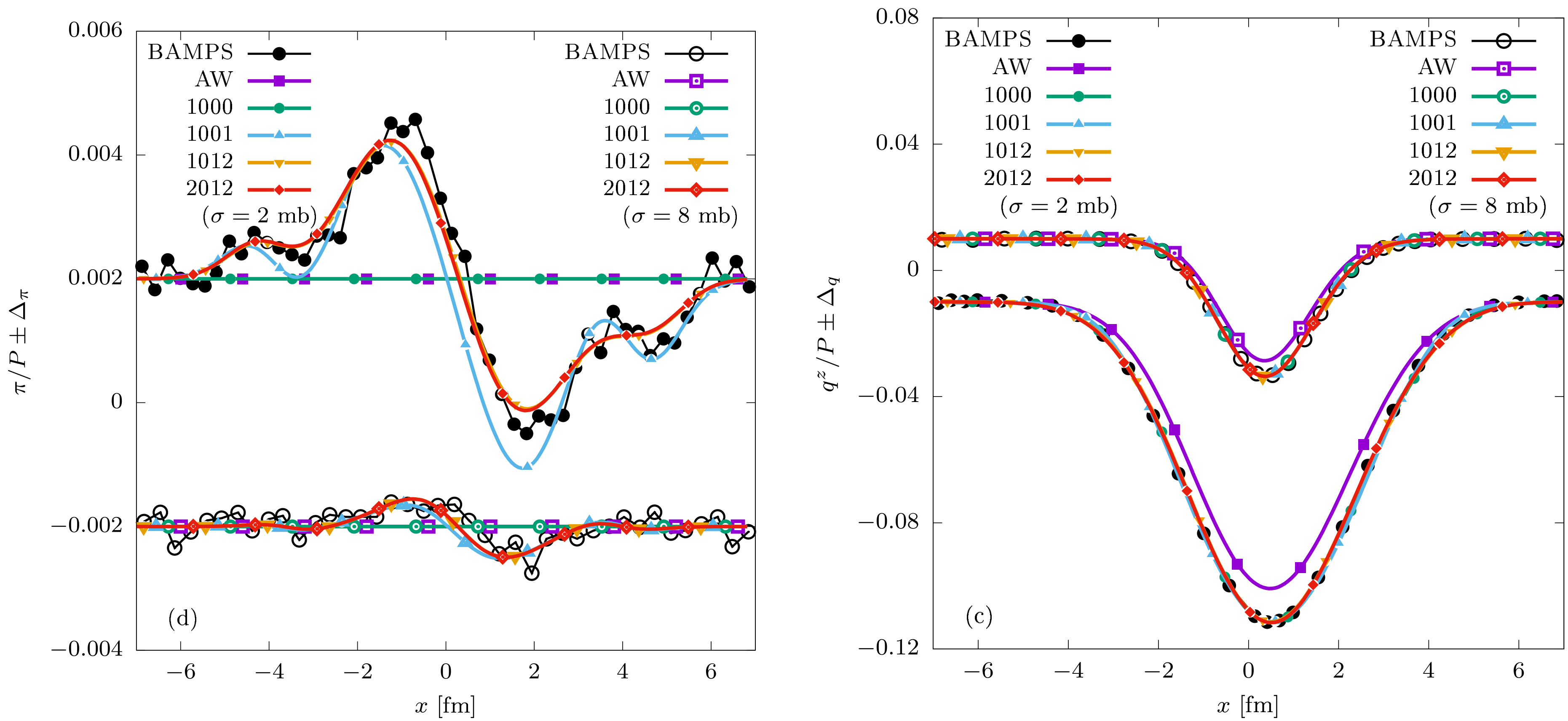
- In the frame of the Sod shock tube, we considered a comparison to BAMPS for hard-sphere interactions.
- Using τ_R to tune η, shear comes out well with AW and Shakhov.
- For diffusion: $1000 \equiv$ first-order Shakhov underestimates peak.
- All high-order Shakhov models perform well!

Heat flow problem: Comparison to BAMPS [DNBMXRG, PRD **89** (2014) 074005]



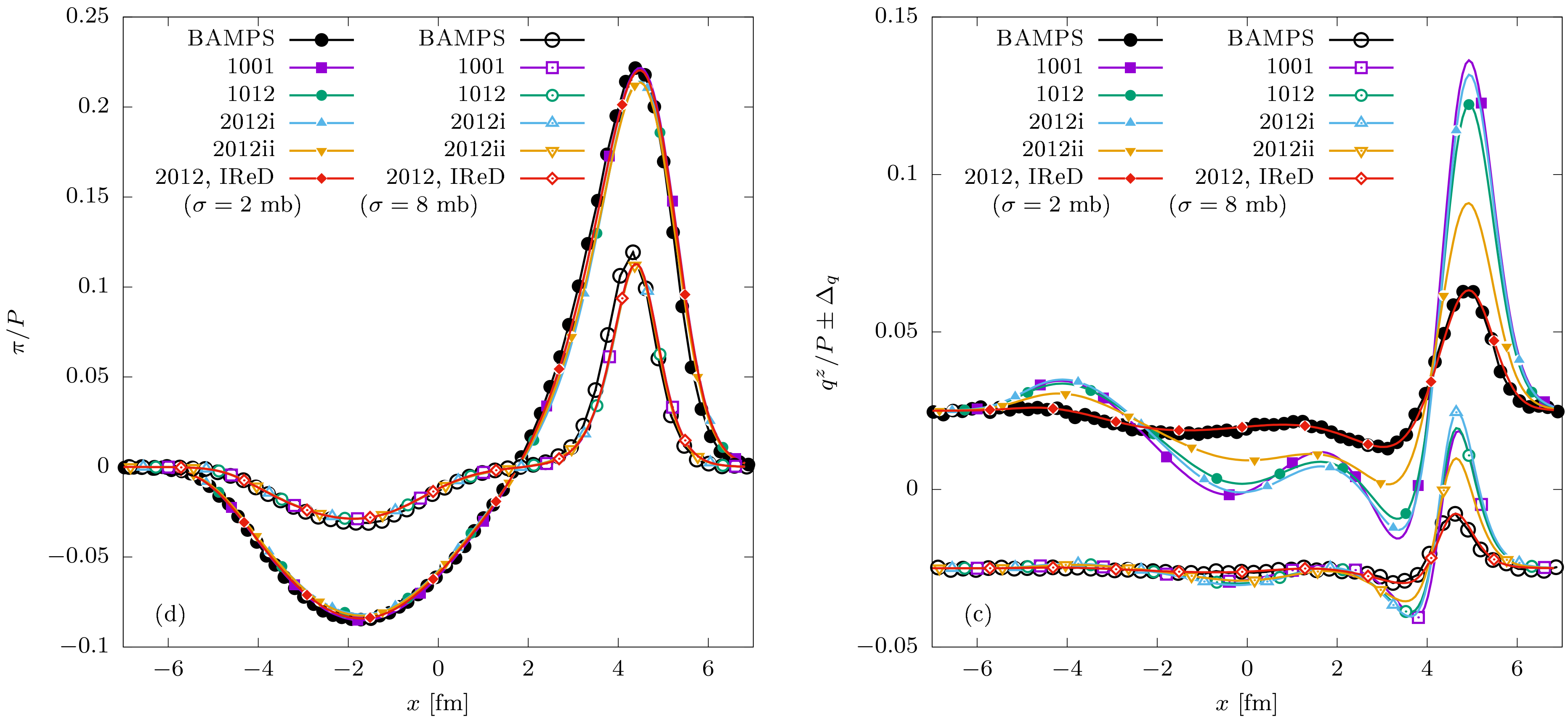
- Case 1: const. initial λ, pressure jump.
- All models recover π/P.
- For q^z, both AW (fixing only η) and 1000 (fixing η and κ) fail.
- All high-order Shahkov models perform well!

Heat flow problem: Comparison to BAMPS [DNBMXRG, PRD **89** (2014) 074005]



- Case 2: cons. initial P, jump in λ.
- AW and 1000 give $\pi/P = 0$; all high-order models recover π/P.
- For q^z, AW is off by $\simeq 10\%$, while 1000 and high-order Shahkov models perform well!

IReD Supremacy: Problem with DNMR



- So far, we used the IReD method for the t.coeffs computation.
- Now we tune the S-model to capture the $O(\mathrm{Re}^{-1}\mathrm{Kn})$ t.coeffs to the DNMR values, ignoring the $O(\mathrm{Kn}^2)$ t.coeffs.
- While π is recovered well, in all S-models the DNMR coefficients lead to wrong results for q^z.

Code availability

- The kinetic equation is solved using a discrete velocity method algorithm based on the relativistic lattice Boltzmann method.
- The source code, run scripts, as well as plotting scripts are available to download from CodeOcean, as follows:
  - $0+1$-D massless Bjorken flow: DOI: 10.24433/CO.5625382.v2
    [VEA et al, Nature Comput. Sci. **2** (2022) 641]
  - $0+1$-D massive Bjorken flow (hydro, aHydro, Boltzmann-RTA): DOI: 10.24433/CO.1942625.v1 [VEA, E. Molnár, D. H. Rischke, arXiv:2311.00351]
  - First-order Shakhov model (Bjorken flow, longitudinal waves): DOI: 10.24433/CO.6267589.v1 [VEA, E. Molnár, arXiv:2311.11603]
  - High-order Shakhov model (Bjorken flow, longitudinal waves, shock waves): DOI: 10.24433/CO.8322373.v1 [VEA, D. Wagner, arXiv:2401.04017]

Kinetic solver: $1+1$-D flows

- For $1+1$-D flows, the kinetic equation reduces to

$$k^t \partial_t f_{\mathbf{k}} + k^z \partial_z f_{\mathbf{k}} = -\frac{E_{\mathbf{k}}}{\tau_R}(f_{\mathbf{k}} - f_{\mathrm{S}\mathbf{k}}). \tag{61}$$

- We parametrize $f_{\mathbf{k}} \equiv f(x^\mu; m_\perp, v^z, \varphi_{\mathbf{k}})$, with

$$\begin{pmatrix} k^t \\ k^z \end{pmatrix} = m_\perp \begin{pmatrix} \cosh y \\ \sinh y \end{pmatrix} = \frac{m_\perp}{\sqrt{1-v_z^2}} \begin{pmatrix} 1 \\ v^z \end{pmatrix}, \qquad \begin{pmatrix} k^x \\ k^y \end{pmatrix} = k_\perp \begin{pmatrix} \cos \varphi_{\mathbf{k}} \\ \sin \varphi_{\mathbf{k}} \end{pmatrix}, \tag{62}$$

where $m_\perp = \sqrt{\mathbf{k}_\perp^2 + m^2}$ is the transverse mass, $y = \tanh^{-1} v^z$ is the rapidity, and $v^z = k^z/k^t$.

- Assuming $u^\mu \partial_\mu = \gamma(\partial_t + \beta^z \partial_z)$, Eq. (61) leads to

$$\partial_t f_{\mathbf{k}} + v^z \partial_z f_{\mathbf{k}} = -\frac{\gamma(1 - \beta^z v^z)}{\tau_R}(f_{\mathbf{k}} - f_{\mathrm{S}\mathbf{k}}). \tag{63}$$

Kinetic solver: Rapidity-based moments

- Going from $\mathbf{k} = (k^x, k^y, k^z)$ to $(m_\perp, v^z, \varphi_k)$ implies:

$$\int \frac{d^3k}{k^0} \to \int_{-1}^{1} \frac{dv^z}{1 - v_z^2} \int_0^{2\pi} d\varphi_k \int_m^\infty dm_\perp m_\perp \,. \tag{64}$$

- The $m_\perp$ and $\varphi_\mathbf{k}$ dofs can be integrated out by introducing *rapidity-based moments*:

$$F_n(v^z) = \frac{g}{(2\pi)^3} \int_0^{2\pi} d\varphi_k \int_m^\infty \frac{dm_\perp \, m_\perp^{n+1}}{(1 - v_z^2)^{(n+2)/2}} f_\mathbf{k}. \tag{65}$$

- For the longitudinal waves and shock waves problems, Eq. (63) can be integrated w.r.t. $m_\perp$ and $\varphi_\mathbf{k}$, leading to

$$\frac{\partial F_n}{\partial t} + v^z \frac{\partial F_n}{\partial z} = -\frac{\gamma(1 - \beta^z v^z)}{\tau}(F_n - F_n^{\mathrm{S}}). \tag{66}$$

- The equation is closed since all required macroscopic quantits. entering $f_{\mathrm{S}\mathbf{k}} \to F_n^{\mathrm{S}}$ can be recovered from F_n:

$$\begin{pmatrix} N_r^t \\ N_r^z \end{pmatrix} = \int_{-1}^{1} dv^z \begin{pmatrix} 1 \\ v^z \end{pmatrix} (u \cdot v)^r F_{r+1} \,, \qquad \begin{pmatrix} T_r^{tt} \\ T_r^{tz} \\ T_r^{zz} \end{pmatrix} = \int_{-1}^{1} dv^z \begin{pmatrix} 1 \\ v^z \\ v_z^2 \end{pmatrix} (u \cdot v)^r F_{r+2} \,. \tag{67}$$

Kinetic solver: Non-conformal Bjorken flow

- Due to the symmetries of Bjorken flow, it is convenient to employ (τ, η), defined by

$$t = \tau \cosh \eta, \qquad z = \tau \sinh \eta. \tag{68}$$

- Due to boost invariance, $f_{\mathbf{k}}$ depends on y and η only through $y - \eta$.
- Then, $f_{\mathbf{k}} \to f(\tau; m_\perp, \varphi_{\mathbf{k}}, v^z)$, where $v^z = \tanh(y - \eta)$ instead of $\tanh y$.
- The kinetic eq. for Bjorken flow becomes:

$$\frac{\partial f_{\mathbf{k}}}{\partial \tau} - \frac{v^z(1 - v_z^2)}{\tau} \frac{\partial f_{\mathbf{k}}}{\partial v^z} = -\frac{1}{\tau_R}(f_{\mathbf{k}} - f_{\mathrm{S}\mathbf{k}}). \tag{69}$$

- Defining again the *rapidity-based moments*,

$$F_n(v^z) = \frac{g}{(2\pi)^3} \int_0^{2\pi} d\varphi_k \int_m^\infty \frac{dm_\perp \, m_\perp^{n+1}}{(1 - v_z^2)^{(n+2)/2}} f_{\mathbf{k}}, \tag{70}$$

one obtains

$$\frac{\partial F_n}{\partial \tau} + \frac{1}{\tau}[1 + (n-1)v_z^2]F_n - \frac{1}{\tau} \frac{\partial [v^z(1 - v_z^2)F_n]}{\partial v^z} = -\frac{1}{\tau_R}(F_n - F_n^{\mathrm{S}}). \tag{71}$$

- The equation is again closed w.r.t. n.

Momentum-space discretization: v^z

- v^z is discretized via the Gauss-Legendre quadrature.
- The continuous functions $F_n(v^z)$ are replaced by

$$F_{n;j} = w_j F_n(v^z_j), \quad w_j = \frac{2(1 - v^2_{z;j})}{[(K+1)P_{K+1}(v^z_j)]^2}, \tag{72}$$

  where v^z_j $(1 \le j \le K)$ satisfy $P_K(v^z_j) = 0$
- The derivative w.r.t. v^z is replaced by the finite sum

$$\left[\frac{\partial[v^z(1 - v_z^2)F_n]}{\partial v^z}\right]_j = \sum_{j'=1}^{K} \mathcal{K}_{j,j'} F_{n;j'}, \tag{73}$$

  where $\mathcal{K}_{j,j'}$ is obtained by projection onto Legendre polynomials:

[VEA, R. Blaga, PRC **98** (2018) 035201]

$$\mathcal{K}_{j,j'} = w_j \sum_{m=1}^{K-3} \frac{m(m+1)(m+2)}{2(2m+3)} P_m(v^z_j) P_{m+2}(v^z_{j'})$$
$$- w_j \sum_{m=1}^{K-1} \frac{m(m+1)}{2} P_m(v^z_j) \left[\frac{(2m+1)P_m(v^z_{j'})}{(2m-1)(2m+3)} + \frac{m-1}{2m-1} P_{m-2}(v^z_{j'})\right]. \tag{74}$$

Conclusions

- Shakhov model generalized for the relativistic Anderson-Witting RTA, allowing ζ, κ and η to be controlled independently.
- Numerical simulations of the Bjorken flow and of sound waves damping confirmed that the model is robust.
- Extending the Shakhov model allows $2^{\rm nd}$-order t. coeffs. to be controlled $\Rightarrow$ agreement with BAMPS in Sod shock tube.
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