Numerical computation of phase diagrams of QCD-inspired models in the large-$N$ limit

Project A03: Inhomogeneous phases at high density

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Outline

1. Introduction
2. GN model in 1+1 dimensions
3. Discretization of the fermionic determinant
4. Numerical results
Goal: calculate phase diagrams of QCD-inspired models in the large-$N$ ($\equiv$ mean field) limit numerically ($N$: number of flavors).

- Particular focus on possibly existing inhomogeneous phases (compute also the shapes of the corresponding condensates).
- (1) **No assumptions, no ansatz** (e.g. no restriction to a chiral density wave etc.).
- (2) **Precise computation** ("no statistical errors").
- (1) + (2) $\rightarrow$ correct and precise field theoretical results.
At the moment we study the Gross-Neveu (GN) model in 1+1 dimensions.
  - Phase diagram analytically known.
  - Ideal to explore and test numerical/lattice field theory methods.

Action:

\[
S = \int d^2x \left( \sum_{j=1}^{N} \bar{\psi}_j \left( \gamma_0 (\partial_0 + \mu) + \gamma_1 \partial_1 \right) \psi_j - \frac{g^2}{2} \left( \sum_{j=1}^{N} \bar{\psi}_j \psi_j \right)^2 \right).
\]

After introducing a scalar field \( \sigma \) (= condensate) and performing the integration over fermionic fields

\[
S_{\text{eff}} = N \left( \frac{1}{2\lambda} \int d^2x \sigma^2 - \ln \left( \text{det} \left( \gamma_0 (\partial_0 + \mu) + \gamma_1 \partial_1 + \sigma \right) \right) \right)
\]

\[
Z = \int D\sigma \ e^{-S_{\text{eff}}},
\]

where \( \lambda = Ng^2 \).
For $N \to \infty$ only that $\sigma$ minimizing $S_{\text{eff}}$ contributes to the partition function.

To study a field theory numerically, the infinite number of degrees of freedom has to be reduced to a finite number $\to$ discretization needed.

- For example Lattice field theory.
- There are other possibilities to discretize, e.g. finite mode discretization, discretization by piecewise polynomials, etc.

Challenges, problems:

- **Discretization of the fermionic determinant** (various problems, e.g. fermion doubling problem, explicit breaking of chiral symmetry, unphysical zero modes, ...).
- **Efficient computation of $\det(Q)$** (after discretization, $Q$ is a large matrix).
Various possibilities tested:

- Expansion in a set of basis functions, e.g. plane waves,

\[ \psi(x, t) \to \sum_{m_t, m_x} c_{m_t, m_x} e^{i(p_{m_t} t + p_{m_x} x)} \]

with \( p_{m_t} = \frac{2\pi (m_t - 1/2)}{L_t}, \) \( p_{m_x} = \frac{2\pi m_x}{L_x} \) and similar for the scalar field \( \sigma \). (requires \( \det(Q) = \det(Q^+) \), not the case for \( \mu_I \neq 0 \) or \( \mu_s \neq 0 \))

- Lattice discretization:
  - Naively discretized fermions. (fermion doubling)

\[ \psi(x, t) \to \psi_{x,t}, \quad \partial_x \psi(x, t) \to \frac{\psi_{x+a,t} - \psi_{x-a,t}}{2a}, \quad ... \]

\( (x, t = 0, a, 2a, \ldots; a: \) lattice spacing).

- Naively discretized with non-symmetric derivatives. (no fermion doubling, but other severe problems)

- Staggered fermions. (fermion doubling still present)

...
Discretization of the fermionic determinant (2)

- Most promising seems to be a combination of two approaches:
  - Plane wave expansion in $t$ direction.
    - **Rather easy analytical simplifications possible**, e.g. $\det(Q)$ factorizes in several smaller determinants.
  - Naive lattice discretization in $x$ direction.
    - $\det(Q) = \det(Q^\dagger)$ **not required**, e.g. computations for non-vanishing isospin and strangeness chemical potential might be possible.
    - **Fermion doubling not a problem**, since we consider the large-$N$ limit ("$2 \times \infty = \infty$ ").

$$\psi(x, t) \rightarrow \sum_m \psi_{x,m} e^{ip_m t}$$

with $p_m = 2\pi(m - 1/2)/L_t$. 
Efficient computation of $\det(Q)$

- $Q$ is a large matrix, e.g. $O(10^5) \times O(10^5)$ entries.
- Needs
  - preparatory analytical simplifications, e.g. to factorize $\det(Q)$,
  - efficient algorithms and codes.
- Work in progress.
- Details are beyond the scope of this presentation.
Numerical results (1)

- Phase diagram with restriction to homogeneous condensate $\sigma$.

\[ N_{T_c} = 8, \ T_c = 16.000, \ L = 384, \ \lambda = 0.480 \]

Numerical computation of phase diagrams of QCD-inspired models in the large-$N$-limit.
Numerical results (2)

- $S_{\text{eff}}(\sigma)$ for homogeneous condensate $\sigma$.
  - Left: far inside the chirally broken phase ($\mu/\sigma_0 = 0.20, \ T = T_c/3$).
  - Center: in the chirally broken phase close to the 1st order phase boundary ($\mu/\sigma_0 = 0.65, \ T = T_c/3$).
  - Right: in the chirally restored phase ($\mu/\sigma_0 = 1.20, \ T = T_c/3$).

- First tests for inhomogeneous condensate $\sigma = \sigma(x)$ successful, i.e. analytically known phase boundaries for inhomogeneous condensate reproduced.