Inhomogeneous phases at high density in QCD-inspired models

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Outline

1. Introduction

2. GN model in $1+1$ dimensions

3. Problems
   - Discretization of the fermionic determinant
   - Efficient computation of $\det(Q)$ and minimization of $S_{\text{eff}}/N$
   - Inhomogeneous phases and finite volume

4. Numerical results
**Long-term goal:** compute the phase diagram of QCD.
- Extremely difficult ...
- ... e.g. “sign problem” in lattice QCD for chemical potential $\mu \neq 0$, computations very challenging/impossible.

**QCD-inspired models in the large-$N$ limit:**
- QCD-inspired = symmetries similar as in QCD, e.g. chiral symmetry
- large-$N$ limit = mean field ($N$: number of flavors)
- Inhomogeneous phases at large $\mu$ and small temperature $T$.
  - inhomogeneous phase = phase with a spatially non-constant order parameter

Cf. e.g.


**Are there inhomogeneous phases in QCD?**
Goals of project A03 "Inhomogeneous phases at high density" of CRC-TR 211 "Strong-interaction matter under extreme conditions":

- Study the phase diagram of various QCD-inspired models (Gross-Neveu (GN), chiral GN, Nambu-Jona-Lasinio (NJL), ...) with particular focus on inhomogeneous phases.

Methods:

- Lattice field theory and related numerical methods (this talk).
- Functional Renormalization Group (talk by A. Königstein).

- Determine the spatial modulation of the condensates (= order parameters) without using specific ansätze (e.g. no restriction to a chiral density wave).

- Are there inhomogeneous phases with 2- or 3-dimensional modulations?

- Phase structure not only with respect to $\mu$ and $T$ but also isospin and strangeness chemical potential $\mu_I$, $\mu_S$.

- Are there inhomogeneous phases at finite $N$?
GN model in 1+1 dimensions (1)

- At the moment we study the GN model in 1+1 dimensions.
  - Phase diagram analytically known.
  - Ideal to explore and test numerical/lattice field theory methods.

- Action:

\[
S = \int d^2x \left( \sum_{j=1}^{N} \bar{\psi}_j \left( \gamma_0 (\partial_0 + \mu) + \gamma_1 \partial_1 \right) \psi_j - \frac{g^2}{2} \left( \sum_{j=1}^{N} \bar{\psi}_j \psi_j \right)^2 \right).
\]

- After introducing a scalar field \( \sigma \) (= condensate) and performing the integration over fermionic fields

\[
S_{\text{eff}} = \mathcal{N} \left( \frac{1}{2\lambda} \int d^2x \sigma^2 - \ln \left( \det (\gamma_0 (\partial_0 + \mu) + \gamma_1 \partial_1 + \sigma) \right) \right)
\]

\[= Q \]

\[
Z = \int D\sigma \, e^{-S_{\text{eff}}},
\]

where \( \lambda = Ng^2 \).
$N \to \infty$: only one field configuration important in $\int D\sigma \, e^{-S_{\text{eff}}}$ (minimum of $S_{\text{eff}}/N$).

For numerical treatment the degrees of freedom have to be reduced to a finite number → finite volume and discretization needed.

- For example lattice field theory.
- There are other possibilities to discretize, e.g. finite mode discretization, discretization by piecewise polynomial functions, etc.


Challenges, problems:

- Discretization of the fermionic determinant (various problems, e.g. fermion doubling problem, explicit breaking of chiral symmetry, unphysical zero modes, ...).
- Efficient computation of $\det(Q)$ and minimization of $S_{\text{eff}}/N$ (after discretization, $Q$ is a large matrix).
- Inhomogeneous phases and finite volume (size of the inhomogeneous structures versus size of the volume).
Various discretizations tested.

Expansion in a set of basis functions, e.g. plane waves,

\[
\psi(x, t) \rightarrow \sum_{mt, mx} c_{mt, mx} e^{i(p_{mt} t + p_{mx} x)}, \quad \sigma(x) \rightarrow \sum_{mx} d_{mx} e^{ip_{mx} x}
\]

with \( p_{mt} = 2\pi(m_t - 1/2)/L_t, p_{mx} = 2\pi m_x/L_x, d_{mx} = (d_{-mx})^* \).


(−) Requires \( \det(Q) = \det(Q^\dagger) \), not the case e.g. for \( \mu_I \neq 0 \) or \( \mu_s \neq 0 \).

- \( \det(Q) \rightarrow \det(\langle f_n | Q | f_{n'} \rangle) \), where \( f_n \) are basis functions, e.g. \( f_{mt, mx} = e^{i(p_{mt} t + p_{mx} x)} \).

- Problem: \( \text{span}\{ f_n \} \neq \text{span}\{ Qf_n \} \), which causes artificially small eigenvalues or zero modes in \( \langle f_n | Q | f_{n'} \rangle \) not present in \( Q \).
  → Wrong and weird results.

- Increasing the number of basis functions does not cure the problem.

- Solution: \( \ln(\det(Q)) \rightarrow (1/2) \ln(\det(Q^\dagger Q)) \) (requires \( \det(Q) = \det(Q^\dagger) \)).

(−) Number of spatial modes in \( \psi(x, t) \) should be larger than number of modes in \( \sigma(x) \).

- \( Q \) depends on \( \sigma(x) \); basis functions representing \( \psi(x, t) \) must be able to resolve more detail for an accurate approximation of \( \det(Q) \).

(+ ) No fermion doubling.

(+ ) Resulting condensates \( \sigma(x) \) are continuous functions.

(When using lattice field theory, \( \sigma(x) \) is represented by a set of points \( \sigma_x \).)
Discretization of the fermionic determinant (2)

- Lattice discretization:
  - Naively discretized fermions.
    \[
    \psi(x, t) \rightarrow \psi_{x,t}, \quad \partial_x \psi(x, t) \rightarrow \frac{\psi_{x+a,t} - \psi_{x-a,t}}{2a}, \quad \ldots
    \]
    \((x, t = 0, a, 2a, \ldots; a: \text{lattice spacing})\).
  - (-) Fermion doubling.
  - Naively discretized with non-symmetric derivatives.
    (-) No fermion doubling, but other severe problems.
  - Staggered fermions.
    (-) Fermion doubling still present.

- Most promising seems to be a combination of two approaches:
  - Plane wave expansion in \(t\) direction.
    (+) Easy analytical simplifications possible, e.g. \(\det(Q)\) factorizes.
  - Naive lattice discretization in \(x\) direction.
    (+) \(\det(Q) = \det(Q^{\dagger})\) not required, \(\mu_I \neq 0\) and \(\mu_S \neq 0\) might be possible.
    (+) Fermion doubling not a problem in the large-\(N\) limit (“\(2 \times \infty = \infty\)”).
    \[
    \psi(x, t) \rightarrow \psi_x(t) = \sum_m \psi_{x,m} e^{ip_m t}, \quad \sigma(x) \rightarrow \sigma_x.
    \]
Efficient computation of det($Q$) and ...

- $Q$ is a large matrix, e.g. $\mathcal{O}(10^5) \times \mathcal{O}(10^5)$ entries.
- Efficient computation of det($Q$) and minimization of $S_{\text{eff}}/N = \ldots - \ln(\det(Q))$ need
  - preparatory analytical simplifications, e.g. to factorize det($Q$),
  - efficient algorithms and codes.
- Work in progress.
- Details are rather technical, beyond the scope of this presentation.
Inhomogeneous phases and finite volume (1)

- Periodic modulation of the inhomogeneous condensate, wave length $\lambda$ depends on $(\mu, T)$ (left figure).
- Extent of the finite volume $L$ typically fixed.
- If $L$ is a multiple of $\lambda$, i.e. $L \approx n\lambda$, $n \in \mathbb{N}_+$ → no particular problems with the finite volume, correct results.
- If $L \approx (n - 1/2)\lambda$, $n \in \mathbb{N}_+$ → modulation of the inhomogeneous condensate does not fit into the finite volume, severely distorted results (right figure, oscillating dashed line).

right figure from [P. de Forcrand and U. Wenger, PoS LATTICE 2006, 152 (2006) [hep-lat/0610117]]
Inhomogeneous phases and finite volume (2)

- Infinite volume phase boundaries can be extracted from finite volume results.
- Phase boundary between inhomogeneous phase and restored phase:
  - Characterized by the appearance/disappearance of negative eigenvalues of the Hessian matrix
    \[ H_{xy} = \frac{\partial}{\partial \sigma_x} \frac{\partial}{\partial \sigma_y} S_{\text{eff}} \bigg|_{\sigma=0} \]
  - Lowest eigenvalue of \( H \) as a function of \( \mu \) oscillates in a finite volume (red curve in left figure):
    - Minima: \( L \approx n\lambda, n \in \mathbb{N}_+ \), identical to the infinite volume result.
    - Maxima: \( L \approx (n - 1/2)\lambda, n \in \mathbb{N}_+ \), significantly different from the infinite volume result.
  - Fitting a smooth curve (e.g. a 2nd order polynomial) from below (green curve in left figure) corresponds to the infinite volume result.
Numerical results (1)

- Plane wave expansion in $t$ direction, lattice discretization in $x$ direction.
- Phase diagram with restriction to homogeneous condensate $\sigma$.

$$N_{T_c} = 8, \ T_c = 16.000, \ L = 384, \ \lambda = 0.480$$

![Graph showing analytical and numerical results for different orderings.](image)
Numerical results (2)

- Plane wave expansion in $t$ direction, lattice discretization in $x$ direction.
- $S_{\text{eff}}(\sigma)$ for homogeneous condensate $\sigma$.
  - Left: far inside the chirally broken phase ($\mu/\sigma_0 = 0.20$, $T = T_c/3$).
  - Center: in the chirally broken phase close to the 1st order phase boundary ($\mu/\sigma_0 = 0.65$, $T = T_c/3$).
  - Right: in the chirally restored phase ($\mu/\sigma_0 = 1.20$, $T = T_c/3$).
First tests for inhomogeneous condensate $\sigma = \sigma(x)$ successful:
- Phase boundary via eigenvalues of the Hessian matrix.
- Numerical results consistent with expectation.
- Work in progress.

[M. Winstel, N. Zorbach, Bachelor of Science theses (ongoing), Goethe University Frankfurt am Main (2018)]
Expansion in piecewise polynomial functions both in $t$ and in $x$ direction.

Phase diagram and condensate $\sigma(x)$.

- **a)** close to the left boundary of the inhomogeneous phase.
- **b)** inside the inhomogeneous phase.

Next steps

- Study the phase diagram of various QCD-inspired models (Gross-Neveu (GN), chiral GN, Nambu-Jona-Lasinio (NJL), ...) with particular focus on inhomogeneous phases.
- Determine the spatial modulation of the condensates (= order parameters) without using specific ansätze (e.g. no restriction to a chiral density wave).
- Extend studies to 3+1 dimensions.
- Are there inhomogeneous phases with 2- or 3-dimensional modulations?
- Phase structure not only with respect to $\mu$ and $T$ but also isospin and strangeness chemical potential $\mu_I, \mu_S$.
- Are there inhomogeneous phases at finite $N$?
  → Lattice field theory simulation of the path integral ...