Exercise 1 [Spin and measurements in quantum mechanics]

In quantum mechanics, the spin $\mathbf{s} = (s_x, s_y, s_z)$ of a particle is described by a set of three operators fulfilling the angular momentum algebra

$$[\hat{s}_j, \hat{s}_k] = i\hbar\epsilon_{jkl}\hat{s}_l,$$

which can be represented by $2 \times 2$ matrices.

(a) Write down a representation of the three spin operators in terms of $2 \times 2$ matrices.

(b) Find the eigenvectors and eigenvalues for each of the three spin operators.

(c) Suppose a system is in the eigenstate of $\hat{s}_x$ with eigenvalue $+\hbar/2$. Calculate the expectation value of a measurement of $s_y$ in this system. What are the possible results of the measurement and what are the corresponding probabilities?

(d) For the same system as in (c), calculate the expectation value for a measurement of spin in the direction $(+1, +1, 0)/\sqrt{2}$, i.e. along an axis at a $45^\circ$ angle w.r.t. the $x$-axis. What are the possible results of the measurement and what are the corresponding probabilities?

(e) For the same system as in (c), we first measure spin in the direction $(+1, +1, 0)/\sqrt{2}$. After that we measure $s_y$. Calculate the expectation value of the latter measurement. What are the possible results of the latter measurement and what are the corresponding probabilities?

(f) Same as (e), but instead of a single measurement at a $45^\circ$ angle, we take $N - 1$ measurements for which the direction of measurement is increased by $90^\circ/N$ for each successive measurement, starting from $0^\circ$ (along the $x$-axis) and ending at $90^\circ$ (along the $y$-axis). Calculate the expectation value of a measurement of $s_y$ taken after these $N - 1$ measurements in the limit $N \to \infty$. What are the possible results of the measurement and what are the corresponding probabilities?

Write down a representation of the operators and states of the angular momentum algebra given above, such that the eigenvalues of $\hat{s}_x, \hat{s}_y, \hat{s}_z$ are $+\hbar, 0$ and $-\hbar$. Represent the operators using
(g) $3 \times 3$ matrices,

(h) suitable combinations of $\mathbf{r}$ and $\nabla$

Which physical systems can be described by these representations?